STEP 2016, Paper 2, Q7 – Solution (2 pages; 24/5/18)

1st part

Let $u = a - x$; then $\int_0^a f(a-x)\,dx = \int_a^0 f(u)(-1)du$

$= \int_0^a f(u)\,du = \int_0^a f(x)\,dx$

(i) By (*),

$I = \int_0^{\frac{\pi}{2}} \frac{sinx}{cosx+sinx}\,dx = \int_0^{\frac{\pi}{2}} \frac{sin(\frac{\pi}{2}-x)}{cos\left(\frac{\pi}{2}-x\right)+sin(\frac{\pi}{2}-x)}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{cosx}{sinx+cosx}\,dx = J, say$

Then $I + J = \int_0^{\frac{\pi}{2}} \frac{sinx+cosx}{cosx+sinx}\,dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

Hence $I = \frac{1}{2}(I+J) = \frac{\pi}{4}$

(ii) $K = \int_0^{\frac{\pi}{4}} \frac{sinx}{cosx+sinx}\,dx = I - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{sinx}{cosx+sinx}\,dx$

Let $u = x - \frac{\pi}{4}$; then $K = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{sin(u+\frac{\pi}{4})}{cos\left(u+\frac{\pi}{4}\right)+sin(u+\frac{\pi}{4})}\,du$

$= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{sinu\left(\frac{1}{\sqrt{2}}\right)+cosu\left(\frac{1}{\sqrt{2}}\right)}{cosu\left(\frac{1}{\sqrt{2}}\right)-sinu\left(\frac{1}{\sqrt{2}}\right)+sinu\left(\frac{1}{\sqrt{2}}\right)+cosu\left(\frac{1}{\sqrt{2}}\right)}\,du$

$= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{sinu+cosu}{2cosu}\,du$

$= \frac{\pi}{4} - \frac{1}{2}[-\ln|cosu|]_0^{\frac{\pi}{4}} - \frac{1}{2}[u]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{8}$

$= \frac{\pi}{8} - \frac{1}{2} ln\sqrt{2}$

(iii) $L = \int_0^{\frac{\pi}{4}} \ln(1 + tanx)\, dx = \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan(\frac{\pi}{4} - x)\right) dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-tanx}{1+tanx}\right) dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+tanx+1-tanx}{1+tanx}\right) dx$

$= \int_0^{\frac{\pi}{4}} ln2 - \ln(1 + tanx)\, dx$

$= [xln2]_0^{\frac{\pi}{4}} - L$

$\Rightarrow L = \frac{1}{2}\left(\frac{\pi}{4}\right) ln2 = \frac{\pi ln2}{8}$

(iv) [The presence of the x in the integrand suggests Parts, provided that $\int \frac{1}{cosx(cosx+sinx)} dx$ can be evaluated, and also the integral of the result, between the limits $0\ and\ \frac{\pi}{4}$]

Consider $\frac{d}{dx}(\ln(1 + tanx))$

$= \frac{1}{1+tanx}(sec^2x) = \frac{1}{cos^2x+sinxcosx} = \frac{1}{cosx(cosx+sinx)}$

So $\int \frac{1}{cosx(cosx+sinx)} dx = \ln(1 + tanx)$

Then, by Parts,

$\int_0^{\frac{\pi}{4}} \frac{x}{cosx(cosx+sinx)} dx = [x \ln(1 + tanx)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1 + tanx)\, dx$

$= \frac{\pi}{4} ln2 - \frac{\pi ln2}{8}$ (from (iii)) $= \frac{\pi ln2}{8}$