STEP 2016, Paper 2, Q13 – Solution (2 pages; 8/6/18)

[3 'show that' results!]

(i) If $X\sim B(16,\frac{1}{2})$ and $Y\sim N(8,4)$,

then $P(X=8)\approx P(7.5<Y<8.5)$

$=P(\frac{7.5-8}{2}<Z<\frac{8.5-8}{2})\ =P(-\frac{1}{4}<Z<\frac{1}{4})$

$=\int_{-\frac{1}{4}}^{\frac{1}{4}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\approx(\frac{1}{\sqrt{2\pi}})(\frac{1}{2})\left(e^{-\frac{1}{2}(0)^2}\right)=\frac{1}{2\sqrt{2\pi}}$

[The approximation here seems quite crude - although it's the midpoint rule with one strip.]

(ii) If $X\sim B(2n,\frac{1}{2})$ and $Y\sim N(n,\frac{n}{2})$,

then $P(X=n)\approx P(n-\frac{1}{2}<Y<n+\frac{1}{2})$

$=P(\frac{-\frac{1}{2}}{\sqrt{\frac{n}{2}}}<Y<\frac{\frac{1}{2}}{\sqrt{\frac{n}{2}}})=\int_{-a}^{a}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$, where $a=\frac{\frac{1}{2}}{\sqrt{\frac{n}{2}}}=\frac{1}{\sqrt{2n}}$

$\approx\left(\frac{1}{\sqrt{2\pi}}\right)(2a)\left(e^{-\frac{1}{2}(0)^2}\right)=\left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\frac{2}{n}}=\frac{1}{\sqrt{n\pi}}$

Also $P(X=n)=\frac{(2n)!}{n!n!}(\frac{1}{2})^{2n}$

So $\frac{(2n)!}{(n!)^2}(\frac{1}{2})^{2n}\approx\frac{1}{\sqrt{n\pi}}$, and hence $(2n)!\approx\frac{2^{2n}(n!)^2}{\sqrt{n\pi}}$, as required.

(iii) For large λ, $Po(\lambda)\sim$approx. $N(\lambda,\lambda)$

so that, if $X\sim Po(\lambda)$ and $Y\sim N(\lambda,\lambda)$, for integer λ:

$P(X=\lambda)\approx P\left(\lambda-\frac{1}{2}<Y<\lambda+\frac{1}{2}\right)=P(\frac{-\frac{1}{2}}{\sqrt{\lambda}}<Z<\frac{\frac{1}{2}}{\sqrt{\lambda}})$

$\approx 2\left(\frac{\frac{1}{2}}{\sqrt{\lambda}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)$, as before

$= \frac{1}{\sqrt{2\pi\lambda}}$

Also $P(X = \lambda) = \frac{e^{-\lambda}\lambda^{\lambda}}{\lambda!}$,

so that $\frac{e^{-\lambda}\lambda^{\lambda}}{\lambda!} = \frac{1}{\sqrt{2\pi\lambda}}$,

Writing $n = \lambda$ then gives $n! \approx e^{-n}n^n\sqrt{2\pi n}$, as required.