STEP 2016, Paper 1, Q2 – Solution (2 pages; 22/5/18)

1st part

$\frac{d}{dx}\{(ax^2+bx+c)ln\left(x+\sqrt{1+x^2}\right)+(dx+e)\sqrt{1+x^2}\}$

$=(2ax+b)ln\left(x+\sqrt{1+x^2}\right)$

$+(ax^2+bx+c)\frac{1+\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)}{x+\sqrt{1+x^2}}$

$+d\sqrt{1+x^2}+(dx+e).\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$

As $\frac{1+\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)}{x+\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}\frac{\sqrt{1+x^2}\left(1+x(1+x^2)^{-\frac{1}{2}}\right)}{x+\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$,

this simplifies to $(2ax+b)ln\left(x+\sqrt{1+x^2}\right)$

$+\frac{1}{\sqrt{1+x^2}}\{ax^2+bx+c+d(1+x^2)+x(dx+e)\}$

$=(2ax+b)ln\left(x+\sqrt{1+x^2}\right)$

$+\frac{1}{\sqrt{1+x^2}}\{(a+2d)x^2+(b+e)x+c+d\}$

(i) Setting $a=0, b=1, d=0, e=-1, c=0$,

$\int \ln(x+\sqrt{1+x^2})\ dx = x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$

(ii) We now want $a+2d=1, b+e=0, c+d=1, a=0, b=0$;

so $d=\frac{1}{2}$, $e=0, c=\frac{1}{2}$

and $\int \sqrt{1+x^2}\ dx=\frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)+\frac{x}{2}\sqrt{1+x^2}$

(iii) We now want $a = \frac{1}{2}, b = 0, d = -\frac{1}{4}, e = 0, c = \frac{1}{4}$

and $\int x ln(x + \sqrt{1 + x^2})\ dx = (\frac{x^2}{2} + \frac{1}{4}) ln(x + \sqrt{1 + x^2})$

$-\frac{x}{4}\sqrt{1 + x^2}$

$= \frac{1}{4}(2x^2 + 1) ln(x + \sqrt{1 + x^2}) - \frac{x}{4}\sqrt{1 + x^2}$