STEP 2016, Paper 1, Q12 – Solution (1 page; 28/5/18)

(i) $P(A=0)P(B>0)+P(A=1)P(B>1)+P(A=2)P(B>2)$

$=\left(\frac{1}{2}\right)^2\left(1-\left(\frac{1}{2}\right)^3\right)+2\left(\frac{1}{2}\right)^2\left(1-\left(\frac{1}{2}\right)^3-3\left(\frac{1}{2}\right)^3\right)+\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$

$=\frac{1}{4}\left\{\frac{7}{8}+\frac{8}{8}+\frac{1}{8}\right\}=\frac{1}{2}$

(ii) $P(A=0)P(B>0)+P(A=1)P(B>1)+P(A=2)P(B>2)$

$+P(A=3)P(B>3)$

$=\left(\frac{1}{2}\right)^3\left(1-\left(\frac{1}{2}\right)^4\right)+3\left(\frac{1}{2}\right)^3\left(1-\left(\frac{1}{2}\right)^4-4\left(\frac{1}{2}\right)^4\right)$

$+3\left(\frac{1}{2}\right)^3\left(4\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^4\right)+\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^4$

$=\frac{1}{8}\left\{\frac{15}{16}+\frac{33}{16}+\frac{15}{16}+\frac{1}{16}\right\}=\frac{1}{2}$

(iii) P(B gets more heads)

= P(B has the same number of heads after n tosses each)

× P(B gets head on n+1 th toss)

+ P(B has more heads after n tosses each) × 1

+ P(B has fewer heads after n tosses each) × 0

$=p_1\left(\frac{1}{2}\right)+p_2$

Now, after n tosses each, either B has more heads, or A has more (with the same probability, by symmetry), or they both have the same number;

so $p_2+p_2+p_1=1$, and hence $p_1\left(\frac{1}{2}\right)+p_2=\frac{1}{2}(p_1+2p_2)=\frac{1}{2}$