STEP 2016, Paper 1, Q11 – Solution (2 pages; 24/2/21)

1st part

The height above the base, $y = h + usin\alpha.t - \frac{1}{2}gt^2$, where t is the time from firing.

The horizontal distance travelled, $X = ucos\alpha.t$

P hits the plain when $y = 0$,

so that, substituting for t, and setting $X = x$,

$h + usin\alpha.\frac{x}{ucos\alpha} - \frac{1}{2}g(\frac{x}{ucos\alpha})^2 = 0$

$\Rightarrow \frac{gx^2}{u^2} = 2hcos^2\alpha + 2xsin\alpha cos\alpha$

$= h\left(1 + cos(2\alpha)\right) + xsin(2\alpha)$, as required (1)

2nd part

Differentiating (1) wrt α:

$\frac{2gx}{u^2}\frac{dx}{d\alpha} = -2hsin(2\alpha) + \frac{dx}{d\alpha}sin(2\alpha) + 2xcos(2\alpha)$

x is maximised (or minimised) when $\frac{dx}{d\alpha} = 0$

$\Rightarrow 0 = -2hsin(2\alpha) + 2xcos(2\alpha)$

$\Rightarrow x = htan(2\alpha)$, as required.

[Note: We could attempt to show that $\frac{d^2x}{d\alpha^2} < 0$, to justify the maximum, but this turns out to be difficult. Normally the Official Sol'ns would require this to be done, but not for this question!]

3rd part

Substituting $x = htan(2\alpha)$ into (1) gives

$\frac{gh^2tan^2(2\alpha)}{u^2} = h\left(1 + cos(2\alpha)\right) + htan(2\alpha)sin(2\alpha)$

$\Rightarrow \frac{ghsin^2(2\alpha)}{u^2cos^2(2\alpha)} = \left(1 + cos(2\alpha)\right) + \frac{sin^2(2\alpha)}{cos(2\alpha)}$ (2)

Writing $A = cos(2\alpha)$, (2) $\Rightarrow \frac{gh\left(1-A^2\right)}{u^2A^2} = 1 + A + \frac{(1-A^2)}{A} = 1 + \frac{1}{A}$

Writing $k = \frac{gh}{u^2}$, $k(1 - A^2) = A^2 + A$,

so that $A^2(k + 1) + A - k = 0$

$\Rightarrow A = \frac{-1\pm\sqrt{1+4(k+1)k}}{2(k+1)} = \frac{-1\pm(2k+1)}{2(k+1)} = \frac{k}{k+1}$ or -1

$A = -1 \Rightarrow \alpha = \frac{\pi}{2}$, which can be rejected (as P is then fired vertically)

So $cos(2\alpha) = \frac{k}{k+1} = \frac{(\frac{gh}{u^2})}{(\frac{gh}{u^2})+1} = \frac{gh}{gh+u^2}$

4th part

The greatest distance between O and the landing point occurs when $x = htan(2\alpha)$ and $cos(2\alpha) = \frac{gh}{gh+u^2}$

This greatest distance is then $\sqrt{h^2 + x^2}$

$= h\sqrt{1 + tan^2(2\alpha)} = hsec(2\alpha) = \frac{h(gh+u^2)}{gh} = h + \frac{u^2}{g}$, as required.