STEP 2016, Paper 1, Q10 – Solution (3 pages; 29/1/21)

(i) **1st part**

[Left to right is taken to be the positive direction]

Collision between A & B:

CoM: $\lambda mu = \lambda mv_A + mv_B;\ \lambda u = \lambda v_A + v_B$ (1)

(with obvious notation [which however needs to be spelt out in the exam])

NLR: $v_B - v_A = eu$ (2)

Then from (1) & (2): $\lambda u = \lambda(v_B - eu\ ) + v_B$,

so that $v_B(\lambda + 1) = \lambda u(1 + e);\ v_B = \frac{\lambda u(1+e)}{\lambda+1}$ (3)

Collision between C & D:

CoM: $m(-u) = mv_C + mv_D;\ -u = v_C + v_D$ (4)

NLR: $-v_C - (-v_D) = eu$ (5)

[v_C & v_D are the velocities from left to right, but v_C will be negative]

Then from (4) & (5): $-u = v_C + (eu + v_C)$,

so that $v_C = -\frac{u}{2}(1 + e)$ (6)

Collision between B & C:

CoM: $mv_B + mv_C = mw_C;\ v_B + v_C = w_C$ (7)

NLR: $w_C - 0 = e(v_B - v_C)$ (8)

Then from (7) & (8): $v_B + v_C = e(v_B - v_C)$

so that $v_C(1+e) = v_B(e-1)$,

and $\frac{v_C}{v_B} = \frac{e-1}{e+1}$

Also, from (3) & (6), $\frac{v_C}{v_B} = \frac{-\frac{u}{2}(1+e)}{\frac{\lambda u(1+e)}{\lambda+1}} = \frac{-(\lambda+1)}{2\lambda}$

Hence $\frac{e-1}{e+1} = \frac{-(\lambda+1)}{2\lambda}$; $2\lambda(1-e) = (\lambda+1)(e+1)$,

so that $e(-2\lambda - \lambda - 1) = \lambda + 1 - 2\lambda$

$\Rightarrow e = \frac{1-\lambda}{-3\lambda-1} = \frac{\lambda-1}{3\lambda+1}$, as required.

2nd part

Thus $e = \frac{\lambda+\frac{1}{3}}{3\lambda+1} - \frac{\frac{4}{3}}{3\lambda+1} = \frac{1}{3} - \frac{4}{3(3\lambda+1)} < \frac{1}{3}$, as required.

(ii) If C and D move towards each other with the same speed (at the end), w_C must be positive and v_D must be negative.

So $w_C = -v_D$

From (4) & (6), $-v_D = u - \frac{u}{2}(1+e) = \frac{u}{2}(1-e)$

And from (7), (3) & (6):

$w_C = v_B + v_C = \frac{\lambda u(1+e)}{\lambda+1} - \frac{u}{2}(1+e) = \frac{u(1+e)}{2(\lambda+1)}\{2\lambda - (\lambda+1)\}$

ie $w_C = \frac{u(1+e)}{2(\lambda+1)}(\lambda-1)$

Then $w_C = -v_D \Rightarrow \frac{u(1+e)}{2(\lambda+1)}(\lambda-1) = \frac{u}{2}(1-e)$

$\Rightarrow (1+e)(\lambda-1) = (\lambda+1)(1-e)$

$\Rightarrow e(\lambda - 1 + \lambda + 1) = \lambda + 1 - (\lambda - 1)$

$\Rightarrow e = \frac{2}{2\lambda} = \frac{1}{\lambda}$

Substituting into the result from (i),

$\frac{1}{\lambda} = \frac{\lambda-1}{3\lambda+1}$; $(3\lambda + 1) = \lambda(\lambda - 1)$

$\Rightarrow \lambda^2 - 4\lambda - 1 = 0$

$\Rightarrow \lambda = \frac{4\pm\sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

Thus, as $\lambda > 1$, $\lambda = 2 + \sqrt{5}$

and $e = \frac{1}{2+\sqrt{5}} = \frac{2-\sqrt{5}}{4-5} = \sqrt{5} - 2$