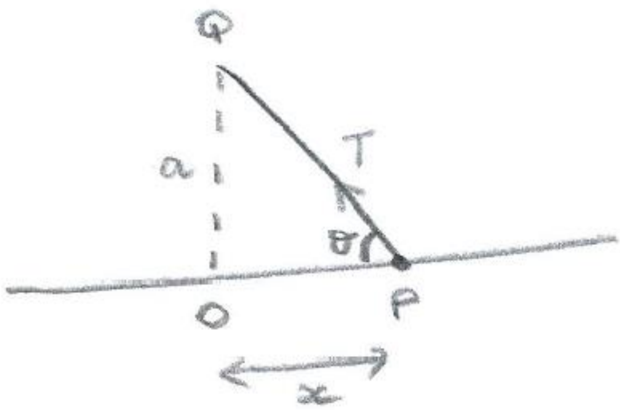
STEP 2015, P3, Q9 - Solution (4 pages; 4/8/20)

1st part



[The question doesn't say so, but we can assume that OQP is in a horizontal plane, otherwise g would appear in the equation of motion.]

By conservation of energy,

$\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{\lambda}{a}\right)e^2$,

where the extension $e = PQ - a = \sqrt{x^2 + a^2} - a$

Hence $v^2 = \dot{x}^2 + k^2\left(\sqrt{x^2 + a^2} - a\right)^2$

and $\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2$ (A)

Alternative (much longer) method

By N2L, $-Tcos\theta = m\ddot{x}$

By Hooke's law, $T = \frac{\lambda e}{a}$,

where the extension $e = PQ - a = \sqrt{x^2 + a^2} - a$

And $cos\theta = \frac{x}{\sqrt{x^2+a^2}}$

So $-\frac{\lambda}{a}\left(\sqrt{x^2+a^2}-a\right)\frac{x}{\sqrt{x^2+a^2}}=m\ddot{x}$

$\Rightarrow \frac{k^2ax}{\sqrt{x^2+a^2}}-k^2x=\ddot{x}$ (B)

[Noting that the LHS can be integrated wrt x]

Also, $\ddot{x}=\frac{d}{dt}(\dot{x})=\frac{d}{dx}(\dot{x}).\dot{x}$

and so, on making the substitution $u=\dot{x}$,

$\int \ddot{x}\ dx=\int\frac{du}{dx}\ u\,dx=\int u\,du=\frac{1}{2}u^2=\frac{1}{2}(\dot{x})^2$

Hence, integrating (B) gives

$\frac{1}{2}k^2a\frac{\sqrt{x^2+a^2}}{\left(\frac{1}{2}\right)}-\frac{1}{2}k^2x^2+C=\frac{1}{2}(\dot{x})^2$

$x=0,\dot{x}=v\Rightarrow k^2a^2+C=\frac{1}{2}v^2$,

so that $C=\frac{1}{2}v^2-k^2a^2$

and $(\dot{x})^2=2k^2a\sqrt{x^2+a^2}-k^2x^2+v^2-2k^2a^2$

$=v^2-k^2(-2a\sqrt{x^2+a^2}+x^2+2a^2)$ (C)

and the given expression $=v^2-k^2(\sqrt{x^2+a^2}-a)^2$ expands to give (C).

2nd part

The greatest & least values of x occur when $\dot{x}=0$

Then $0=v^2-k^2\left(\sqrt{x^2+a^2}-a\right)^2$,

so that $v=k(\sqrt{x^2+a^2}-a)$ (as $v,k>0$)

and $x^2+a^2=\left(\frac{v}{k}+a\right)^2$, so that $x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2va}{k}}$

(with the least value occurring at $x = -x_0$)

3rd part

Applying N2L, as in the alternative method of the 1st part, gives (B) above: $\frac{k^2ax}{\sqrt{x^2+a^2}} - k^2x = \ddot{x}$

When $x = x_0$, $\ddot{x} = \frac{k^2a}{\frac{v}{k}+a}\sqrt{\frac{v^2}{k^2} + \frac{2va}{k}} - k^2\sqrt{\frac{v^2}{k^2} + \frac{2va}{k}}$,

as ${x_0}^2 + a^2 = \left(\frac{v}{k} + a\right)^2$, from the working to the 2nd part.

So $\ddot{x} = (\frac{k^2a}{\frac{v}{k}+a} - k^2)\sqrt{\frac{v^2}{k^2} + \frac{2va}{k}}$

$= k\left(\frac{a-(\frac{v}{k}+a)}{\frac{v}{k}+a}\right)\sqrt{v^2 + 2vak}$

$= \frac{-kv}{(v+ak)}\sqrt{v^2 + 2vak}$

4th part

From (A), $\dot{x} = \sqrt{v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2}$,

so that $\frac{dt}{dx} = \frac{1}{\sqrt{v^2-k^2\left(\sqrt{x^2+a^2}-a\right)^2}}$

and $\int_0^{\frac{T}{4}} dt = \int_0^{x_0} \frac{1}{\sqrt{v^2-k^2\left(\sqrt{x^2+a^2}-a\right)^2}}\, dx$,

by the symmetry of the motion.

Hence $T = \frac{4}{v}\int_0^{x_0} \frac{1}{\sqrt{1-\frac{k^2}{v^2}\left(\sqrt{x^2+a^2}-a\right)^2}}\ dx$

5th part

Let $u^2 = \frac{k}{v}\left(\sqrt{x^2+a^2}-a\right)$, where $u > 0$,

so that $\frac{vu^2}{k} + a = \sqrt{x^2+a^2}$

and $x^2 = \left(\frac{vu^2}{k}+a\right)^2 - a^2$

$= \frac{v^2u^4}{k^2} + \frac{2vu^2a}{k}$

$= \frac{2vu^2a}{k}\left(\frac{vu^2}{2ka}+1\right)$

Then, as $\frac{v}{ka} \approx 0$, $x \approx u\sqrt{\frac{2va}{k}}$

and $\frac{dx}{du} \approx \sqrt{\frac{2va}{k}}$

Then, when $x = 0, u = 0$,

and when $x = x_0$ (so that $x^2 + a^2 = \left(\frac{v}{k}+a\right)^2$),

$u^2 = \frac{k}{v}\left(\frac{v}{k}\right) = 1$, so that $u = 1$

Then $T \approx \frac{4}{v}\int_0^1 \frac{1}{\sqrt{1-u^4}}\ \sqrt{\frac{2va}{k}}du$

$\approx \sqrt{\frac{32a}{kv}}\int_0^1 \frac{1}{\sqrt{1-u^4}}\ du$, as required.