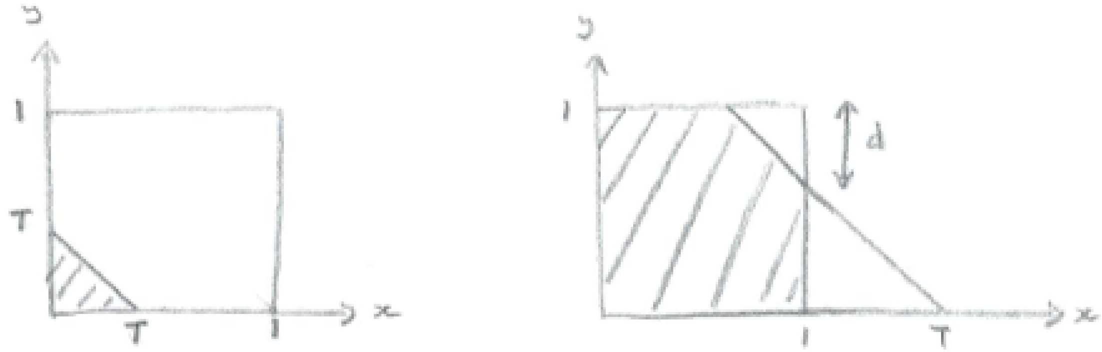
STEP 2015, P3, Q13 - Solution (5 pages; 9/3/21)

(i) 1st part

All possible points (x, y) are contained in the square of base 1, and are equally likely.



Referring to the diagrams,

the CDF, $P(X + Y \leq T) = \frac{\frac{1}{2}T^2}{1} = \frac{1}{2}T^2$ for $0 \leq T \leq 1$,

$= \frac{1-\frac{1}{2}d^2}{1}$ for $1 \leq T \leq 2$

$= 0$ otherwise

And $d = 1 - (T - 1) = 2 - T$,

so that $1 - \frac{1}{2}d^2 = 1 - \frac{1}{2}(2 - T)^2 = 2T - \frac{1}{2}T^2 - 1$

and $P(X + Y \leq T) = 0$ for $T < 0$

$= \frac{1}{2}T^2$ for $0 \leq T \leq 1$,

$= 2T - \frac{1}{2}T^2 - 1$ for $1 \leq T \leq 2$

$= 1$ for $T > 2$

[On reflection, a lower case letter probably would have been better here; say v instead of T, as upper case letters tend to be reserved for random variables. (T was being used to avoid confusion with the t in the next part.)]

2nd part

Hence $F(t) = P\left(\frac{1}{X+Y} \leq t\right) = P\left(X + Y \geq \frac{1}{t}\right)$

$= 1 - P\left(X + Y < \frac{1}{t}\right)$

$= 1 - \frac{1}{2}(\frac{1}{t})^2$ for $0 \leq \frac{1}{t} \leq 1$ (as $P\left(X + Y < \frac{1}{t}\right) = P\left(X + Y \leq \frac{1}{t}\right)$)

$= 1 - \{2(\frac{1}{t}) - \frac{1}{2}\left(\frac{1}{t}\right)^2 - 1\}$ for $1 \leq \frac{1}{t} \leq 2$

$= 1$ for $\frac{1}{t} > 2$

ie $1 - \frac{1}{2t^2}$ for $1 \leq t < \infty$

$= 2 - \frac{2}{t} + \frac{1}{2t^2}$ for $\frac{1}{2} \leq t \leq 1$

$= 0$ otherwise

Then $f(t) = F'(t) = 2t^{-2} - t^{-3}$ for $\frac{1}{2} \leq t \leq 1$

$= t^{-3}$ for $1 \leq t < \infty$

$= 0$ otherwise, as required.

3rd part

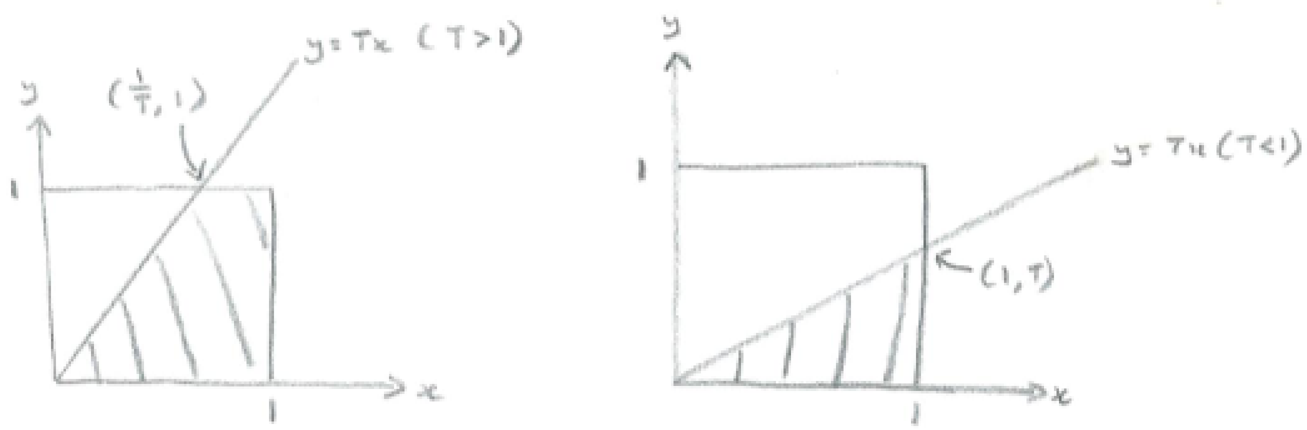
$E\left(\frac{1}{X+Y}\right) = \int_{\frac{1}{2}}^{1} t(2t^{-2} - t^{-3})dt + \int_{1}^{\infty} t.t^{-3}dt$

$= \left[2lnt + \frac{1}{t}\right]_{\frac{1}{2}}^{1} + [-\frac{1}{t}]_{1}^{\infty}$

$= (0 + 1) - (-2ln2 + 2) + (0) - (-1)$

$= 2ln2$

(ii) **1st part**



Referring to the diagrams,

$P\left(\frac{Y}{X} \leq T\right) = \frac{1}{2}(1)T = \frac{T}{2}$ for $0 \leq T \leq 1$

$= 1 - \frac{1}{2}(1)\left(\frac{1}{T}\right) = 1 - \frac{1}{2T}$ for $T > 1$

2nd part

$P\left(\frac{X}{X+Y} \leq t\right) = P(\frac{1}{1+\frac{Y}{X}}) \leq t) = P(1 + \frac{Y}{X} \geq \frac{1}{t}) = P(\frac{Y}{X} \geq \frac{1}{t} - 1)$

$= 1 - P(\frac{Y}{X} \leq \frac{1}{t} - 1)$

From the 1st part, this

$= 1 - \frac{(\frac{1}{t}-1)}{2}$ for $0 \leq \frac{1}{t} - 1 \leq 1$

& $\frac{1}{2(\frac{1}{t}-1)}$ for $\frac{1}{t} - 1 > 1$

ie $\frac{2t-(1-t)}{2t}$ for $1 \leq \frac{1}{t} \leq 2$

& $\frac{t}{2(1-t)}$ for $\frac{1}{t} > 2$

ie $\frac{t}{2(1-t)}$ for $0 \leq t < \frac{1}{2}$

& $\frac{3t-1}{2t}$ for $\frac{1}{2} \leq t \leq 1$

(noting that $\frac{X}{X+Y}$ cannot be negative or greater than 1)

Differentiating wrt t, the pdf of $\frac{X}{X+Y}$ is

$\frac{1}{2}\frac{d}{dt}\left(\frac{t-1}{1-t} + \frac{1}{1-t}\right) = \frac{1}{2}(1-t)^{-2}$ for $0 \leq t < \frac{1}{2}$,

$\frac{d}{dt}\left(\frac{3}{2} - \frac{1}{2t}\right) = \frac{1}{2}t^{-2}$ for $\frac{1}{2} \leq t \leq 1$,

and 0 otherwise

3rd part

$E\left(\frac{X}{X+Y}\right) + E\left(\frac{Y}{X+Y}\right) = E\left(\frac{X+Y}{X+Y}\right) = E(1) = 1$

By symmetry, $E\left(\frac{X}{X+Y}\right) = E\left(\frac{Y}{X+Y}\right)$, so that $E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$

4th part

$$E\left(\frac{X}{X+Y}\right) = \int_0^{\frac{1}{2}} t.\frac{1}{2}(1-t)^{-2}dt + \int_{\frac{1}{2}}^{1} t.\frac{1}{2}t^{-2}dt$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{t-1}{(1-t)^2} + \frac{1}{(1-t)^2}\ dt + \frac{1}{2}[lnt]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{2}[\ln(1-t) + (1-t)^{-1}]_0^{\frac{1}{2}} + \frac{1}{2}(0 + ln2)$$

$$= \frac{1}{2}(-ln2 + 2) - \frac{1}{2}(0+1) + \frac{1}{2}ln2$$

$$= \frac{1}{2}$$