STEP 2015, P3, Q12 - Solution (6 pages; 5/8/20)

(i) 1st part

$G(x) = \sum_{r=0}^{5} P(R_1 = r)\, x^r$

$= \frac{1}{6}\sum_{r=0}^{5} x^r$

$= \frac{1}{6} \cdot \frac{x^6-1}{x-1}$

2nd part

pgf of $R_2 = \sum_{r=0}^{5} P(R_2 = r)\, x^r$

The sample space diagram for R_2 is:

	1	2	3	4	5	6
1	2	3	4	5	0	1
2	3	4	5	0	1	2
3	4	5	0	1	2	3
4	5	0	1	2	3	4
5	0	1	2	3	4	5
6	1	2	3	4	5	0

So $P(R_2 = r) = \frac{6}{36} = \frac{1}{6}$, and hence the pgf of R_2 is also $G(x)$.

Alternative method (much longer)

The pgf of R_2 can be obtained from $[G(x)]^2$, by combining the powers of x mod 6 (so that eg the coefficient of x^8 is added to the coefficient of x^2):

$[G(x)]^2 = \left(\frac{1}{6} \cdot \frac{x^6-1}{x-1}\right)^2$

$= \frac{1}{36}(x^5 + x^4 + \cdots + 1)^2$

$= \frac{1}{36}(x^{10} + x^8 + \cdots + 1 + 2x^9 + 2x^8 + \cdots + 2x^5 + 2x^7 + \cdots + 2x^4$

$+2x^5 + \cdots + 2x^3 + 2x^3 + 2x^2 + 2x)$

$= \frac{1}{36}(x^{10} + 2x^9 + 3x^8 + 4x^7 + 5x^6 + 6x^5 + 5x^4 + 4x^3 + 3x^2 +$
$2x + 1)$

$= \frac{1}{36}\{(x^{10} + 5x^4) + (2x^9 + 4x^3) + (3x^8 + 3x^2) + (4x^7 + 2x) +$
$(5x^6 + 1) + 6x^5\}$

and all the coefficients of the combined powers are seen to be $\frac{1}{6}$.

3rd part

To find the pgf of R_3, the remainder from R_1 (corresponding to the 3rd throw of the die) can be combined with the remainder from R_2, mod 6. As the pgf of R_2 is $G(x)$, this gives the same result as when the pgf of R_2 is derived.

In the same way, the pgfs of $R_3, R_4, \ldots, R_n$ are also $G(x)$.

And so $\mathrm{P}(S_n$ is divisible by 6$) = \mathrm{P}(S_1$ is divisible by 6$) = \frac{1}{6}$

(ii) **1st part**

$G_1(x) = \sum_{r=0}^{4} P(T_1 = r)\, x^r$

$= \frac{1}{6} + \frac{2}{6}x + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4$

(both 1 & 6 have a remainder of 1, so that the coefficient of x is $\frac{2}{6}$)

$= \frac{1}{6}(x + y)$, where $y = 1 + x + x^2 + x^3 + x^4$

[using the notation adopted later on in the question]

2nd part

The sample space diagram for T_2 is:

	1	2	3	4	5	6
1	2	3	4	0	1	2
2	3	4	0	1	2	3
3	4	0	1	2	3	4
4	0	1	2	3	4	0
5	1	2	3	4	0	1
6	2	3	4	0	1	2

So $G_2(x) = \frac{7}{36} + \frac{7}{36}x + \frac{8}{36}x^2 + \frac{7}{36}x^3 + \frac{7}{36}x^4$

$= \frac{7}{36}y + \frac{1}{36}x^2$

$= \frac{1}{36}(x^2 + 7y)$, as required.

[The official sol'n takes the longer route of using $[G_1(x)]^2$]

3rd part

[The pgf of T_n could in theory be obtained from $[G_1(x)]^n$ by combining the powers of x mod 5. However, this is unlikely to be manageable algebraically for general n. From an exam technique point of view, the method adopted in the mark scheme is not attractive (even though it is likely to be the only feasible one!): (a) it involves a lot of work (b) the result for $G_n(x)$ is not given [though the result for P(S_n is divisible by 5) provides a partial check] (c) there is no suggestion to use induction in the question, so it could be the case that a quicker method is being overlooked.]

Consider $G_3(x)$

[There are two possible ways of deriving this using pgfs: (a) finding $[G_1(x)]^3$, and combining the powers of x mod 5, and (b) finding $G_2(x).G_1(x)$, and again combining the powers of x mod 5. This should involve less work than (a).]

$G_3(x)$ can be obtained from $G_2(x).G_1(x)$, by combining the powers of x mod 5.

$G_2(x).G_1(x) = \frac{1}{36}(x^2 + 7y).\frac{1}{6}(x + y)$

$= \frac{1}{6^3}(x^3 + 7x(1 + x + x^2 + x^3 + x^4)$

$+x^2(1 + x + x^2 + x^3 + x^4) + 7(1 + x + x^2 + x^3 + x^4)^2)$

$= \frac{1}{6^3}\{7 + x(7 + 14) + x^2(7 + 1 + 7 + 14)$

$+x^3(1 + 7 + 1 + 14 + 14) + x^4(7 + 1 + 7 + 14 + 14)$

$+x^5(7 + 1 + 14 + 14) + x^6(1 + 7 + 14) + x^7(14) + x^8(7)\}$

Combining powers of x mod 5 gives

$\frac{1}{6^3}\{(7 + 36) + (21 + 22)x + (29 + 14)x^2 + (37 + 7)x^3 + 43x^4\}$

$= \frac{1}{6^3}(x^3 + 43y)$

The proposition:

$G_n(x) = \frac{1}{6^n}(x^{n(mod\ 5)} + [1 + 6 + 6^2 + \cdots + 6^{n-1}]y)$

or $\frac{1}{6^n}\left(x^{n(mod\ 5)} + \frac{y(6^n-1)}{5}\right)$, can be investigated by induction.

If it is true, then P(S_n is divisible by 5) = constant term of $G_n(x)$

If n is not divisible by 5,

then constant term $= \frac{1}{6^n}[1 + 6 + 6^2 + \cdots + 6^{n-1}]$

$= \frac{1}{6^n} \cdot \frac{6^n - 1}{5}$

$= \frac{1}{5}(1 - \frac{1}{6^n})$, as required.

If n is divisible by 5,

then constant term $= \frac{1}{5}\left(1 - \frac{1}{6^n}\right) + \frac{1}{6^n}$

$= \frac{1}{5}\left(1 + \frac{4}{6^n}\right)$

First of all, the proposition is true for $n = 1$.

Now suppose that $G_k(x)$ is true.

Then $G_{k+1}(x)$ is obtained from $G_k(x). G_1(x)$, by combining the powers of x mod 5.

$G_k(x). G_1(x)$

$= \frac{1}{6^k}\left(x^{k(mod\ 5)} + [1 + 6 + 6^2 + \cdots + 6^{k-1}]y\right). \frac{1}{6}(x + y)$

$= \frac{1}{6^{k+1}}\{x^{k+1(mod\ 5)} + y[x^k + (x + y)(1 + 6 + 6^2 + \cdots + 6^{k-1})\}$

Now $yx^k \equiv y$ (iro powers of x mod 5)

and $yx(1 + 6 + 6^2 + \cdots + 6^{k-1}) \equiv y(1 + 6 + 6^2 + \cdots + 6^{k-1})$

Also $y^2 \equiv y + y + \cdots + y = 5y$,

so that $y^2(1 + 6 + 6^2 + \cdots + 6^{k-1}) \equiv 5y(1 + 6 + 6^2 + \cdots + 6^{k-1})$

So $y[x^k + (x + y)(1 + 6 + 6^2 + \cdots + 6^{k-1})$

$\equiv y + y(1 + 6 + 6^2 + \cdots + 6^{k-1}) + 5y(1 + 6 + 6^2 + \cdots + 6^{k-1})$

$\equiv y + 6y(1 + 6 + 6^2 + \cdots + 6^{k-1})$

$\equiv y(1 + 6 + 6^2 + \cdots + 6^k)$

and so $G_{k+1}(x) = \frac{1}{6^{k+1}}\{x^{k+1(mod\ 5)} + (1 + 6 + 6^2 + \cdots + 6^k)y\}$

Thus, if $G_k(x)$ is true, then $G_{k+1}(x)$ is true.

As $G_n(x)$ is true for $n = 1$, it is therefore true for $n = 2, 3, \ldots$, and hence all positive integers, by the principle of induction.

[Arguably there is far too much work to be done here in the time available.]