STEP 2015, Paper 2, Q6 Solution (2 pages; 25/1/21)

(i) **1st part**

$sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{1}{\left(cos\frac{\pi}{4}cos\frac{x}{2}+sin\frac{\pi}{4}sin\frac{x}{2}\right)^2}=\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}$

$=\frac{2}{cos^2\left(\frac{\pi}{2}\right)+sin^2\left(\frac{\pi}{2}\right)+2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}=\frac{2}{1+sinx}$, as required

2nd part

$\int\frac{1}{1+sinx}dx=\frac{1}{2}\int sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx=\frac{1}{2}(-2)\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c$

$=-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c$

(ii) **1st part**

With $y=\pi-x$, $I=\int_0^\pi xf(sinx)dx$

$=\int_\pi^0(\pi-y)f(\sin(\pi-y))(-1)dy$

$=\pi\int_0^\pi f(siny)dy-\int_0^\pi yf(siny)dy$

$\Rightarrow 2I=\pi\int_0^\pi f(siny)dy$,

so that $I=\frac{\pi}{2}\int_0^\pi f(sinx)dx$, as required

2nd part

Hence $\int_0^\pi\frac{x}{1+sinx}\,dx=\frac{\pi}{2}\int_0^\pi\frac{1}{1+sinx}\,dx$

$=\frac{\pi}{2}\left[-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]_0^\pi$, from (i)

$=\frac{\pi}{2}(1+1)=\pi$

(iii) [$\int_0^{\pi} \frac{2x^3-3\pi x^2}{(1+sinx)^2} dx$ can be reduced to integrals of the form

$A\int_0^{\pi} \frac{1}{(1+sinx)^2} dx$, using (ii)]

Consider $J = \int_0^{\pi} \frac{1}{(1+sinx)^2} dx = \frac{1}{4}\int_0^{\pi} sec^4(\frac{\pi}{4} - \frac{x}{2})dx$, from the 1st part of (i)

[As $sec^4\theta = sec^2\theta(tan^2\theta + 1)$, and $\int sec^2\theta\, dy = tan\theta$:]

Let $y = \tan(\frac{\pi}{4} - \frac{x}{2})$, so that $dy = sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\left(-\frac{1}{2}\right)dx$

Then $J = \frac{1}{4}\int_1^{-1}(y^2 + 1)(-2)dy$

$= \frac{1}{2}[\frac{1}{3}y^3 + y]_{-1}^{1}$

$= \frac{1}{2}\left(\frac{1}{3} + 1\right) - \frac{1}{2}(-\frac{1}{3} - 1) = \frac{4}{3}$

Now $\int_0^{\pi} \frac{2x^3-3\pi x^2}{(1+sinx)^2} dx = \left[2\left(\frac{\pi}{2}\right)^3 - 3\pi\left(\frac{\pi}{2}\right)^2\right]J$,

by repeated application of the 1st part of (ii).

$= \pi^3\left(\frac{1}{4} - \frac{3}{4}\right)\left(\frac{4}{3}\right) = -\frac{2}{3}\pi^3$