STEP 2015, Paper 2, Q12 Solution (4 pages; 28/1/21)

(i) [An idea for tackling this can be obtained from part (iii), where probabilities are conditioned on the first two outcomes. You might expect though that part (i) will be easier than part (iii), so it would be worth looking for a simpler approach. However, it's debatable whether the first method below is simpler than the alternative method based on the idea from (iii).]

$P(A) = P(TT)P(A|TT) + P(HT)P(A|HT)$

$+P(TH)P(A|TH) + P(HH)P(A|HH)$

$= \frac{1}{4}\{P(A|TT) + P(A|HT) + P(A|TH) + P(A|HH)\}$ (*)

Consider $P(A|TT)$:

A needs an H before any progress can be made. Suppose that the sequence TT...TH occurs (noting that B will not have won in the meantime). Then, if TT...THH follows, B will have won, and if

TT...THT follows, A will be back at square 1. Clearly, the sequence THH will occur eventually, and so only B can win.

Thus, $P(A|TT) = 0$

Consider $P(A|HT)$: The same reasoning applies,

and $P(A|HT) = 0$

Consider $P(A|TH)$: If the sequence THH follows, then B will have won, and if the sequence THT follows, then A cannot win, as

$P(A|HT) = 0$

Consider $P(A|HH)$:

B needs a T before any progress can be made, but as soon as a T occurs, A will win first. As a T will occur eventually, only A can win, and so $P(A|HH) = 1$.

So, from (*), $P(A) = \frac{1}{4}(0 + 0 + 0 + 1) = \frac{1}{4}$, as required.

Alternative method

$P(A) = P(TT)P(A|TT) + P(HT)P(A|HT)$

$+P(TH)P(A|TH) + P(HH)P(A|HH)$ (*)

Now, $P(A|TT) = P(H)P(A|TH) + P(T)P(A|TT)$

$P(A|HT) = P(H)P(A|TH) + P(T)P(A|TT)$

$P(A|TH) = P(H).0 + P(T)P(A|HT)$

$P(A|HH) = P(H)P(A|HH) + P(T).1$

The last 4 eq'ns simplify to:

$P(A|TT) = \frac{1}{2}P(A|TH) + \frac{1}{2}P(A|TT)$ (1)

$P(A|HT) = \frac{1}{2}P(A|TH) + \frac{1}{2}P(A|TT)$ (2)

$P(A|TH) = \frac{1}{2}P(A|HT)$ (3)

$P(A|HH) = \frac{1}{2}P(A|HH) + \frac{1}{2}$, so that $\boldsymbol{P(A|HH) = 1}$

$(1) \Rightarrow P(A|TT) = P(A|TH)$ $(1')$

$(1')$ & $(2) \Rightarrow P(A|HT) = \frac{1}{2}P(A|TH) + \frac{1}{2}P(A|TH)$

$= P(A|TH)$ (4)

Then, from (3), $\boldsymbol{P(A|TH) = P(A|HT) = 0}$

And, from (1′), $\boldsymbol{P(A|TT) = 0}$

Hence, from (*), $P(A) = \frac{1}{4}(0 + 0 + 0 + 1) = \frac{1}{4}$

(ii) As before,

$P(A) = \frac{1}{4}\{P(A|TT) + P(A|HT) + P(A|TH) + P(A|HH)\}$

As $P(A|TT), P(A|HT) \& P(A|TH)$ are all zero when just A and B are playing, they must be zero with all 4 players playing (the only difference is that C or D may win before B can win).

For $P(A|HH)$, note that A will win before B or C (by the same reasoning as before), because B and C can make no progress until a T has occurred, when A will have won. Also, A will win before D, as D requires two Ts.

So $P(A|HH) = 0$ again, and $P(A) = \frac{1}{4}$ again.

By symmetry, $P(C) = \frac{1}{4}$ as well (swapping H & T).

Also $P(D) = P(B)$, by symmetry, for the same reason.

So $\frac{1}{4} + \frac{1}{4} + 2P(B) = 1$, and hence $P(D) = P(B) = \frac{1}{4}$

Thus each player has a probability of winning of $\frac{1}{4}$.

(iii) **1st part**

$P(C|TT) = P(H).1 + P(T)P(C|TT)$,

so that $P(C|TT)\left(1 - \frac{1}{2}\right) = \frac{1}{2}$, and hence $P(C|TT) = 1$

2nd part

$p = P(C|HT) = P(H)P(C|TH) + P(T)P(C|TT)$

$= \frac{1}{2}q + \frac{1}{2}\ .1$; ie $p = \frac{1}{2} + \frac{1}{2}q$, as required

3rd part

$P(C) = P(TT)P(C|TT) + P(HT)P(C|HT)$

$+P(TH)P(C|TH) + P(HH)P(C|HH)$

$= \frac{1}{4}.1 + \frac{1}{4}p + \frac{1}{4}q + \frac{1}{4}P(C|HH)$

Also, $q = P(C|TH) = P(H).0 + P(T)P(C|HT) = \frac{1}{2}p$

So $p = \frac{1}{2} + \frac{1}{2}q = \frac{1}{2} + \frac{1}{2}(\frac{1}{2}p)$,

and hence $\frac{3}{4}p = \frac{1}{2}$, and $p = \frac{2}{3}$; $q = \frac{1}{3}$

Also $P(C|HH) = P(C)$ (as neither B nor C have made any progress),

so that $P(C) = \frac{1}{4} + \frac{1}{4}\left(\frac{2}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}P(C)$,

and hence $\frac{3}{4}P(C) = \frac{1}{2}$, and $P(C) = \frac{2}{3}$