**STEP 2015, Paper 2, Q11 Solution** (2 pages; 10/3/21)

(i) **1st part**

Coordinates of $A$ are : $(x - acos\theta, asin\theta)$

**2nd part**

Differentiating wrt $t$, velocity of $A$ is $(v + asin\theta.\dot{\theta}, acos\theta.\dot{\theta})$,

or $(v + a\dot{\theta}sin\theta, a\dot{\theta}cos\theta)$, as required.

[Note that $v = \dot{x}$ is not necessarily constant.]

(ii) **1st part**

By symmetry, the velocity of B is $(v + a\dot{\theta}sin\theta, -a\dot{\theta}cos\theta)$.

By conservation of linear momentum,

$$m\begin{pmatrix} u \\ 0 \end{pmatrix} = m\begin{pmatrix} v \\ 0 \end{pmatrix} + m\begin{pmatrix} v + a\dot{\theta}sin\theta \\ a\dot{\theta}cos\theta \end{pmatrix} + m\begin{pmatrix} v + a\dot{\theta}sin\theta \\ -a\dot{\theta}cos\theta \end{pmatrix},$$

so that $u = 3v + 2a\dot{\theta}sin\theta$, as required.

**2nd part**

By conservation of energy,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + 2.\frac{1}{2}m\{(v + a\dot{\theta}sin\theta)^2 + (a\dot{\theta}cos\theta)^2\},$$

so that $u^2 = v^2 + 2\{(v + a\dot{\theta}sin\theta)^2 + (a\dot{\theta}cos\theta)^2\}$

$= 3v^2 + 4va\dot{\theta}sin\theta + 2a^2(\dot{\theta})^2$

From the 1st part, $v = \frac{1}{3}(u - 2a\dot{\theta}sin\theta)$, and so

$$u^2 = \frac{1}{3}(u - 2a\dot{\theta}sin\theta)^2 + \frac{4}{3}(u - 2a\dot{\theta}sin\theta)a\dot{\theta}sin\theta + 2a^2(\dot{\theta})^2$$

Writing $k = a\dot{\theta}$, for the moment,

$3u^2 = u^2 - 4uksin\theta + 4k^2sin^2\theta + 4(u - 2ksin\theta)ksin\theta + 6k^2$

$\Rightarrow 2u^2 = 4k^2sin^2\theta - 8k^2sin^2\theta + 6k^2$

$\Rightarrow u^2 = (3 - 2sin^2\theta)k^2$ ,

so that $(\dot{\theta})^2 = \frac{k^2}{a^2} = \frac{u^2}{a^2(3-2sin^2\theta)}$, as required.

(iii) As no energy is lost in the collision, the results of (ii) continue to apply. When $\theta = 0$, after $A$ and $C$ have collided, $\theta$ is increasing, and thereafter $\dot{\theta}$ is never zero (from the 2nd result of (ii)). So $\theta$ continues to increase, and therefore the 2nd collision between $A$ and $C$ occurs when $\theta = \pi$.

(iv) From the 1st result of (ii), when $v = 0$,

$(a\dot{\theta})^2 = \frac{u^2}{4sin^2\theta}$, and substituting into the 2nd result gives

$4sin^2\theta = 3 - 2sin^2\theta$,

so that $sin^2\theta = \frac{1}{2}$, giving $sin\theta = \pm\frac{1}{\sqrt{2}}$,

and hence $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, as $A$ remains above the $x$-axis.

Now, $A$ will oscillate between $\theta = 0$ & $\theta = \pi$, and $\dot{\theta}$ will be negative as $A$ returns to $\theta = 0$. In that situation, the 1st result of (ii) $\Rightarrow 3v = u - 2a\dot{\theta}sin\theta > u$, so that $v$ won't be zero.