STEP 2015, Paper 1, Q9 Solution (2 pages; 21/2/20)

1st part

Let $T(\alpha)$ be the time from firing when a bullet hits the ground.

Then $usin\alpha T(\alpha) - g[T(\alpha)]^2 = 0 \Rightarrow T(\alpha) = \frac{2usin\alpha}{g}$ (as $T(\alpha) \neq 0$).

A bullet fired at time t hits the ground at time

$T_1(\alpha) = t + \frac{2usin\alpha}{g}$, where $\alpha = \frac{\pi}{3} - \lambda t$,

so that $T_1(\alpha) = \frac{\frac{\pi}{3}-\alpha}{\lambda} + \frac{2usin\alpha}{g}$

In the case of the last bullet to hit the ground, $T_1(\alpha)$ is a maximum; ie $\frac{d}{d\alpha}T_1(\alpha) = 0$,

so that $-\frac{1}{\lambda} + \frac{2u}{g}cos\alpha = 0 \Rightarrow cos\alpha = \frac{g}{2u\lambda} = k$

The range of this bullet is $(ucos\alpha)T(\alpha) = \frac{2u^2 sin\alpha cos\alpha}{g}$

$= \frac{2u^2}{g}\sqrt{1-k^2}.k$, as required.

This result is valid for $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{3}$; ie $\frac{1}{2} \leq k \leq \frac{\sqrt{3}}{2}$.

2nd part

When $k < \frac{1}{2}$, $T_1(\alpha)$ is maximised (when $cos\alpha = k$) at $\alpha = \alpha_1 > \frac{\pi}{3}$

Consider $\frac{d}{d\alpha}T_1(\alpha) = -\frac{1}{\lambda} + \frac{2u}{g}cos\alpha = -\frac{1}{\lambda} + \frac{1}{\lambda k}cos\alpha$

$= \frac{1}{\lambda k}(cos\alpha - k)$

For $\alpha < \alpha_1$, $cos\alpha > cos\alpha_1 = k$, so that $\frac{d}{d\alpha}T_1(\alpha) > 0$

Thus $T_1(\alpha)$ is an increasing function, and so its greatest value for $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{3}$ occurs when $\alpha = \frac{\pi}{3}$; ie when the range is

$$\frac{2u^2\sin(\frac{\pi}{3})cos(\frac{\pi}{3})}{g} = \frac{2u^2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{g} = \frac{u^2\sqrt{3}}{2g}$$