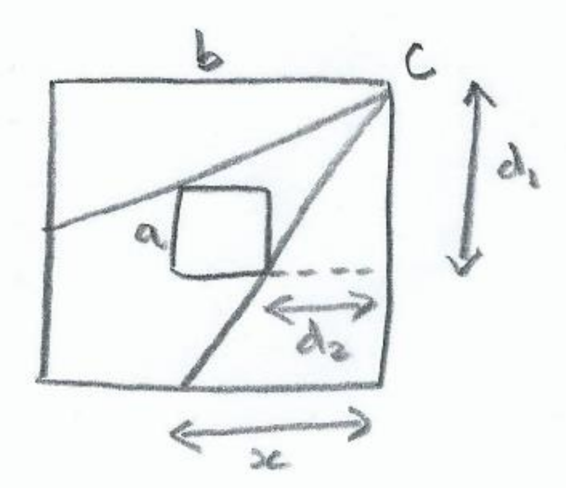
STEP 2015, P1, Q3 - Solution (4 pages; 31/7/20)

Guard standing at a corner



Referring to the diagram above, the total length of the perimeter that the guard can see is $p_C = 2b + 2x$

$d_1 = a + \frac{1}{2}(b-a) = \frac{1}{2}(a+b)$

And $d_2 = \frac{1}{2}(b-a)$

Then $x = d_2 . \frac{b}{d_1}$ (by similar triangles)

so that $p_C = 2b + (b-a)\frac{b}{\frac{1}{2}(a+b)}$

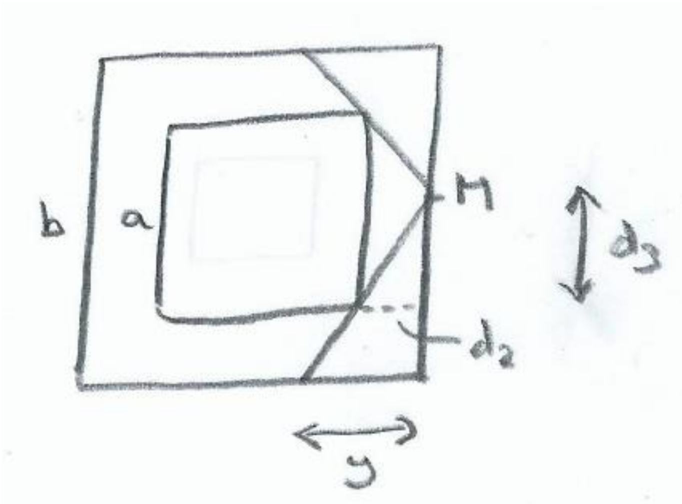
$= \frac{2b(a+b)+2(b-a)b}{a+b}$

$= \frac{4b^2}{a+b}$

Guard standing at the middle of the wall

There are 2 possible scenarios

Scenario A



Referring to the diagram above, the total length of the perimeter that the guard can see is:

$p_M = b + 2y$

And $y = d_2.\frac{\left(\frac{b}{2}\right)}{d_3}$

$= \frac{1}{2}(b - a)\frac{\left(\frac{b}{2}\right)}{\left(\frac{a}{2}\right)}$

$= \frac{(b-a)b}{2a}$

Then $p_M = b + 2y = b + \frac{(b-a)b}{a}$

$= \frac{b}{a}(a + [b - a])$

$= \frac{b^2}{a}$

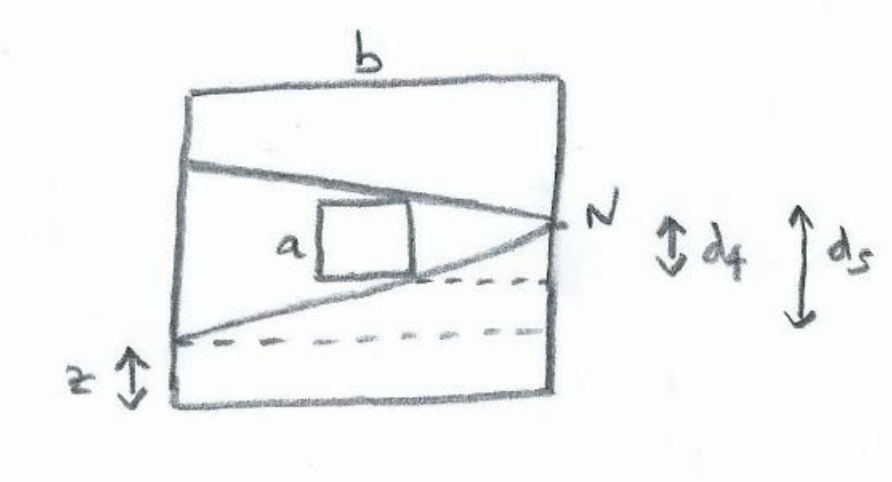
This scenario applies when $y \leq b$;

ie when $\frac{(b-a)b}{2a} \leq b$

$\Leftrightarrow b - a \leq 2a$

$\Leftrightarrow b \leq 3a$

Scenario B



Referring to the diagram above, the total length of the perimeter that the guard can see is: $p_N = 3b + 2z$

Now, $\frac{d_5}{d_4} = \frac{b}{d_2}$

$\Rightarrow d_5 = \frac{b\left(\frac{a}{2}\right)}{\frac{1}{2}(b-a)} = \frac{ab}{b-a}$

and $z = \frac{b}{2} - d_5$

$= \frac{b}{2} - \frac{ab}{b-a}$

$= \frac{b(b-a)-2ab}{2(b-a)}$

$= \frac{b(b-3a)}{2(b-a)}$

Then $p_N = 3b + \frac{b(b-3a)}{b-a}$

$= \frac{3b(b-a)+b(b-3a)}{b-a}$

$= \frac{b(4b-6a)}{b-a}$

$= \frac{2b(2b-3a)}{b-a}$

Conclusion

When $b < 3a$ (so that scenario A applies),

$p_C - p_M = \frac{4b^2}{a+b} - \frac{b^2}{a}$

$= \frac{b^2}{a(a+b)}(4a - [a + b])$

$= \frac{b^2(3a-b)}{a(a+b)} > 0$

so that the corner is better

When $b > 3a$ (so that scenario B applies),

$p_C - p_N = \frac{4b^2}{a+b} - \frac{2b(2b-3a)}{b-a}$

$= \frac{4b^2(b-a)-2b(2b-3a)(a+b)}{b^2-a^2}$

$= \frac{2b(2b^2-2ab-[2ab+2b^2-3a^2-3ab])}{b^2-a^2}$

$= \frac{2b(-ab+3a^2)}{b^2-a^2}$

$= \frac{2ab(3a-b)}{b^2-a^2} < 0$

so that the middle of the wall is better

When $b = 3a$, either of scenarios A and B can be applied, and the two positions are equally good.