STEP 2015, P1, Q13 - Solution (3 pages; 15/3/21)

[Part (i) arguably encourages the use of conditional probabilities

for some of the other parts. The official sol'ns do condition on when the 1st 6 arises, but without involving conditional probabilities.]

(i) $P(A) = \left(\frac{5}{6}\right)^{n-1} . \frac{1}{6}$

(ii) Event B is the same as "a 5 occurs before a 6", so $P(A) = \frac{1}{2}$

[Alternatively:

$P(B) = \sum_{r=2}^{\infty}\{P(1st\ 6\ arises\ on\ rth\ throw)[1 - P(no\ 5s\ arise\ in\ 1st\ r-1\ throws$

$|no\ 6s\ arise\ in\ the\ 1st\ r-1\ throws)]\}$

[Noting that, if we know that a 6 has not arisen in the 1st $r-1$ throws, then at each throw there are only 5 possible (and equally likely) outcomes.]

$= \sum_{r=2}^{\infty}\left(\frac{5}{6}\right)^{r-1} . \frac{1}{6}(1 - \left(\frac{4}{5}\right)^{r-1})$

$= \frac{5}{6} . \frac{1}{6} . \frac{1}{1-\frac{5}{6}} - \frac{1}{6}\sum_{r=2}^{\infty}\left(\frac{4}{6}\right)^{r-1}$

$= \frac{5}{6} - \frac{1}{6} . \frac{4}{6} . \frac{1}{1-\frac{4}{6}}$

$= \frac{5}{6} - \frac{4}{12} = \frac{1}{2}$]

(iii) Event $B \cap C$ is the same as "of the 3 numbers: 4,5 & 6, the 6 arises last", and so $P(B \cap C) = \frac{1}{3}$

(iv) $P(D) =$

$\sum_{r=2}^{\infty}\{P(1st\ 6\ arises\ on\ rth\ throw)P(exactly\ one\ 5\ arises\ in\ the\ 1st$

$r-1\ throws|no\ 6s\ arise\ in\ the\ 1st\ r-1\ throws)\}$

$= \sum_{r=2}^{\infty}\left(\frac{5}{6}\right)^{r-1} . \frac{1}{6} . \binom{r-1}{1}(\frac{1}{5})(\frac{4}{5})^{r-2}$

$= \frac{5}{6} . \frac{1}{6} . \frac{1}{5} \sum_{r=2}^{\infty}(r-1)\left(\frac{4}{6}\right)^{r-2}$

$= \frac{1}{36}\sum_{R=0}^{\infty}(R+1)\left(\frac{2}{3}\right)^{R}$ (where $R = r-2$)

$= \frac{1}{36}\sum_{R=1}^{\infty} R\left(\frac{2}{3}\right)^{R} + \frac{1}{36}\sum_{R=0}^{\infty}\left(\frac{2}{3}\right)^{R}$ (*)

Consider $\sum_{R=1}^{\infty} Ra^{R} = a\frac{d}{da}\sum_{R=1}^{\infty} a^{R} = a\frac{d}{da}(\frac{a}{1-a})$ (when $|a| < 1$)

$= a . \frac{(1-a)-a(-1)}{(1-a)^2} = \frac{a}{(1-a)^2}$

Then (*) $= \frac{1}{36} . \frac{(\frac{2}{3})}{(\frac{1}{9})} + \frac{1}{36} . \frac{1}{1-\frac{2}{3}} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$

(v) $P(D \cup E) = P(D) + P(E) - P(D \cap E)$

And $P(D \cap E) =$
$\sum_{r=2}^{\infty}\{P(1st\ 6\ arises\ on\ rth\ throw)P(exactly\ one\ 4\ \&\ one\ 5\ arise\ in\ the$

$1st\ r-1\ throws|no\ 6s\ arise\ in\ the\ 1st\ r-1\ throws)\}$

$= \sum_{r=3}^{\infty}\left(\frac{5}{6}\right)^{r-1} . \frac{1}{6} . {}^{r-1}P_2(\frac{1}{5})(\frac{1}{5})(\frac{3}{5})^{r-3}$,

as there are ${}^{r-1}P_2 = \binom{r-1}{2}$ 2! ways of placing the 4 & 5 in the 1st
$r-1$ throws,

$= \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}(2!)(\frac{1}{5})(\frac{1}{5})\sum_{r=3}^{\infty} \binom{r-1}{2}(\frac{3}{6})^{r-3}$

$= 2\left(\frac{1}{6}\right)^3 \sum_{R=0}^{\infty} \binom{R+2}{2}(\frac{1}{2})^R$, where now $R = r - 3$ (**)

From the suggestion in the question,

$(1-x)^{-3} = 1 + 3x + 6x^2 + \cdots + \frac{(r+2)!}{r!2!}x^r + \cdots$

so that (**) $= 2\left(\frac{1}{6}\right)^3 (1-\frac{1}{2})^{-3} = 2\left(\frac{2}{6}\right)^3 = \frac{2}{27}$

From (iii), $P(D) = \frac{1}{4}$, and $P(E) = \frac{1}{4}$ in the same way

So $P(D \cup E) = \frac{1}{4} + \frac{1}{4} - \frac{2}{27} = \frac{27-4}{54} = \frac{23}{54}$