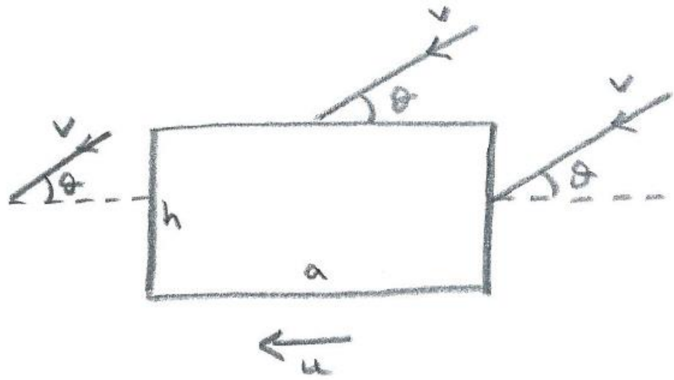
STEP 2015, P1, Q10 - Solution (4 pages; 3/8/20)

1st part



The rain has velocity $\begin{pmatrix} vcos\theta \\ vsin\theta \end{pmatrix}$, where the positive directions are to the North and downwards.

When $u = 0$, the amounts of rain hitting different parts of the bus in unit time are as follows:

back: $kbhvcos\theta$, where b is the width of the bus, and k is the density of the rain per unit volume

roof: $kbavsin\theta$

front: 0

Thus the total amount is proportional to $hvcos\theta + avsin\theta$

2nd part

When $0 < u \leq vcos\theta$, these amounts become

back: $kbh(vcos\theta - u)$ (based on the velocity of the rain relative to the bus)

roof: $kbavsin\theta$

front: 0

And when $u > vcos\theta$, the amounts become back: 0

roof: $kbavsin\theta$

front: $kbh(u - vcos\theta)$ (as the bus catches up with the rain)

[In the official Hints & Sol'ns, they say "while the rain hits the front of the bus as before, but with $u - vcos\theta$ instead of $vcos\theta$". This should read: "while the rain hits the front of the bus, instead of the back, and with $u - vcos\theta$ instead of $vcos\theta$"]

Thus the total amount (in any case) is proportional to

$h|vcos\theta - u| + avsin\theta$, as required.

3rd part

The total amount of rain over the whole journey is proportional to

$= \{h|vcos\theta - u| + avsin\theta\}.\frac{d}{u}$, where d is the distance travelled

ie it is proportional to $h\left|\frac{vcos\theta}{u} - 1\right| + \frac{avsin\theta}{u}$

Let $f(u) = h\left(\frac{vcos\theta}{u} - 1\right) + \frac{avsin\theta}{u}$, for $u \leq vcos\theta$,

and $g(u) = h\left(1 - \frac{vcos\theta}{u}\right) + \frac{avsin\theta}{u}$, for $u \geq vcos\theta$

Then $f'(u) = -\frac{v(hcos\theta + asin\theta)}{u^2}$

Thus $f(u)$ decreases as u increases.

Also, $g'(u) = \frac{v(hcos\theta - asin\theta)}{u^2}$

So, if $w < vcos\theta$ (so that $u < vcos\theta$ and $f(u)$ applies), then

$f(u)$ is minimised when $u = w$

If instead $w \geq vcos\theta$, then $f(u)$ has its least value when u is as large as possible; ie when $u = vcos\theta$, when $g(u)$ has the same value as $f(u)$. When $asin\theta > hcos\theta$, $g(u)$ decreases as u

increases, and so the least value of $g(u)$ (which is $\leq$ the least value of $f(u)$) occurs for the largest possible value of u; ie w - as required.

4th part

If $w > vcos\theta$ and $asin\theta < hcos\theta$, then $g(u)$ increases beyond

$u = vcos\theta$, and so the least value of $f(u)$ or $g(u)$ occurs at

$u = vcos\theta$

5th part

If $asin\theta = hcos\theta$, then $g(u)$ is constant, and so the amount of rain is minimised for any $u \geq vcos\theta$

[The official Hints & Sol'ns say "we may as well take $u = w$", but the driver may not want to travel at the maximum speed!]

6th part

For the return journey, the total amount of rain per unit time is proportional to $h(vcos\theta + u) + avsin\theta$, where the 1st term is due to rain hitting the front, and the 2nd term is from rain hitting the roof.

And so the total amount of rain for the whole (return) journey is

proportional to $h(\frac{vcos\theta}{u} + 1) + \frac{avsin\theta}{u}$ (for all u).

This differs from $f(u)$ above by a constant, and hence is minimised when u is as large as possible; ie when $u = w$.

[The official Hints & Sol'ns say "simply replace θ by $180 - \theta$".

Note that, in this case, $cos\theta$ will be negative, and so u is always

$\geq vcos\theta$, so that $g(u)$ applies, rather than $f(u)$. Sometimes the examiners insist on a convincing explanation as to why the result can be extended to values of θ outside the original range (arguably it isn't obvious that it can), so this strategy might be risky for other questions.]