Admissions Testing Service

STEP Mark Scheme 2015

Mathematics

STEP 9465/9470/9475

October 2015

Test

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners and shows the main valid approaches to each question. It is recognised that there may be other approaches and if a different approach was taken in the exam these were marked accordingly after discussion by the marking team. These adaptations are not recorded here.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

The Admissions Testing Service will not enter into any discussion or correspondence in connection with this mark scheme.

More information about STEP can be found at:
www.stepmathematics.org.uk

Contents

STEP Mathematics (9465, 9470, 9475)

SI-2015/Q1

(i) $y = e^x(2x-1)(x-2)$ **B1** Correct factorisation of quadratic term (or formula, etc.)

$(\frac{1}{2}, 0)$ & $(2, 0)$ **B1** Noted or shown on sketch

$\frac{dy}{dx} = e^x(2x^2 - x - 3)$ **M1** Derivative attempted and equated to zero for TPs

$= e^x(2x-3)(x+1)$

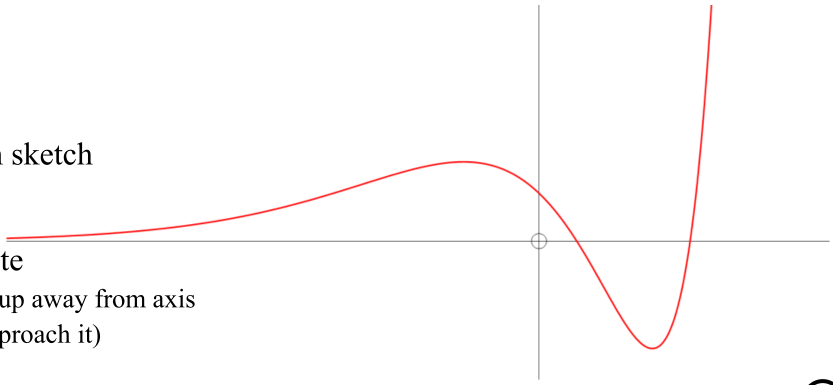
$(\frac{3}{2}, -e^{1.5})$ & $(-1, 9e^{-1})$ **A1 A1** Noted or shown on sketch

(if y-coords. missing, allow one A1 for 2 correct x-coords.)

G1 Generally correct shape

G1 for (0, 2) noted or shown on sketch

G1 for negative-x-axis asymptote
(penalise curves that clearly turn up away from axis
or that do not actually seem to approach it)



⑧

Give **M1** for either 0, 1, 2 or 3 solutions *OR* clear indication they know these arise from where a horizontal line meets the curve (e.g. by a line on their diagram) – implied by any correct answer(s)

Then $y = k$ has

NO solutions for $k < -e^{1.5}$	**A1**
ONE solution for $k = -e^{1.5}$ and $k > 9e^{-1}$	**A1 A1**
TWO solutions for $-e^{1.5} < k \le 0$ and $k = 9e^{-1}$	**A1 A1**
THREE solutions for $0 < k < 9e^{-1}$	**A1**

FT from their y-coords.of the Max. &Min. points.

⑦

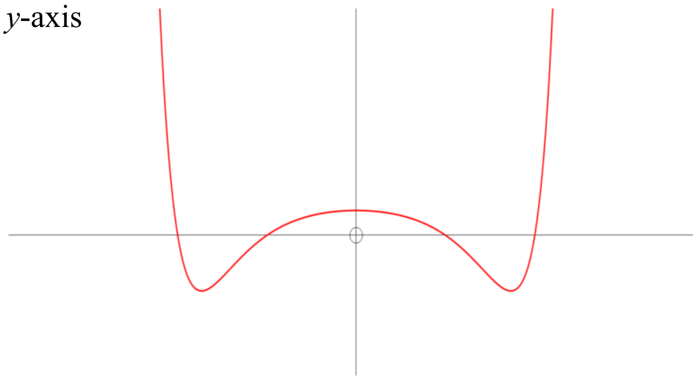
(ii) **G1** Any curve clearly symmetric in y-axis

G1 Shape correct

G1 A Max. TP at (0, 2) **FT**

G1 Min. TPs at $(\pm\sqrt{\frac{3}{2}}, -e^{1.5})$ **FT**

G1 Zeroes at $x = \pm\sqrt{\frac{1}{2}}, \pm\sqrt{2}$ **FT**



⑤

SI-2015/Q2

(i) **M1** Use of cos($A - B$) formula with $A = 60^o$, $B = 45^o$ *OR* $A = 45^o$, $B = 30^o$
or $2\cos^2 15^o - 1$ etc.

A1 Exact trig.values used (visibly) to gain $\cos 15^o = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ *legitimately* (**Given Answer**)

M1 Similar method *OR* $\sin = +\sqrt{1-\cos^2}$ (as 15^o is acute, no requirement to justify +vesq.rt.)

A1 $\sin 15^o = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (however *legitimately* obtained) ④

(ii) **M1** Use of cos($A + B$) formula *and* double-angle formulae *OR* de Moivre'sThm. (etc.)

A1 $\cos 3\alpha \equiv 4\cos^3\alpha - 3\cos\alpha$

A1 Justifying/noting that $x = \cos\alpha$ is thus a root of $4x^3 - 3x - \cos 3\alpha = 0$

M1 For serious attempt to factorise $4(x^3 - c^3) - 3(x - c)$ as linear × quadratic factors
or via *Vieta's Theorem* (roots/coefficients)

A1 $(x - c)\{4(x^2 + cx + c^2) - 3\}$

M1 Solving $4x^2 + 4cx + (4c^2 - 3) = 0$ **FT** their quadratic factor

Remaining roots are $x = \frac{1}{2}\left(-c \pm \sqrt{c^2 - (4c^2 - 3)}\right)$

M1 Use of $s = \sqrt{1 - c^2}$ to simplify sq.rt. term

A1 $x = \frac{1}{2}\left(-\cos\alpha \pm \sqrt{3}\sin\alpha\right)$

⑧

(iii) **M1** $\dfrac{1}{2}y^3 - \dfrac{3}{2}y - \dfrac{\sqrt{2}}{2} = 0$

A1 $4\left(\dfrac{1}{2}y\right)^3 - 3\left(\dfrac{1}{2}y\right) - \dfrac{\sqrt{2}}{2} = 0$

M1 $\cos 3\alpha = \dfrac{\sqrt{2}}{2} = \cos 45^o$

A1 $\Rightarrow \alpha = 15^o$

M1 $\dfrac{1}{2}y = \cos\alpha$, $\frac{1}{2}\left(-\cos\alpha + \sqrt{3}\sin\alpha\right)$, $\frac{1}{2}\left(-\cos\alpha - \sqrt{3}\sin\alpha\right)$ with their α

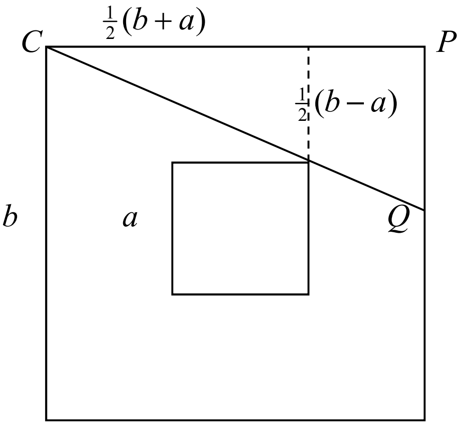
A1 $y = 2\cos 15^o = \dfrac{\sqrt{3}+1}{\sqrt{2}}$

A1 $\sqrt{3}\sin 15^o - \cos 15^o = -\dfrac{\sqrt{3}-1}{\sqrt{2}}$

A1 $-\sqrt{3}\sin 15^o - \cos 15^o = -\sqrt{2}$

⑧

SI-2015/Q3



B1 For correct lengths in smallerΔ

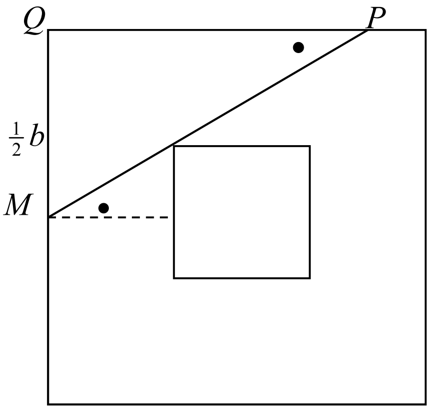
M1 By similar Δs (*OR* trig.*OR*coord.geom.)

A1 $\dfrac{PQ}{b} = \dfrac{\frac{1}{2}(b-a)}{\frac{1}{2}(b+a)} \Rightarrow PQ = \dfrac{b(b-a)}{b+a}$

M1 so a guard at a corner can see $2(b+PQ)$

A1 $= \dfrac{4b^2}{b+a}$ (might be given as all but $\dfrac{4ba}{b+a}$ or as a fraction of the perimeter)

⑤



Lengths $\frac{1}{2}a$ and $\frac{1}{2}(b-a)$ in smaller Δ

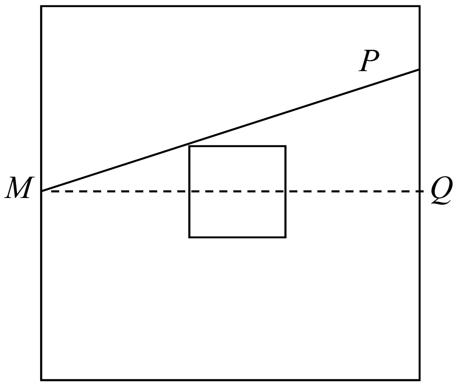
M1 By similar Δs (*OR* trig.*OR*coord.geom.)

A1 $\dfrac{\frac{1}{2}b}{PQ} = \dfrac{\frac{1}{2}a}{\frac{1}{2}(b-a)} \Rightarrow PQ = \dfrac{b(b-a)}{2a}$

M1 so a guard at a midpoint can see $b + 2PQ$

A1 $= \dfrac{b^2}{a}$ (might be given as all but $\dfrac{b(4a-b)}{a}$ or as a fraction of the perimeter)

④



Lengths $\frac{1}{2}a$ and $\frac{1}{2}(b-a)$ in smaller Δ

M1 By similar Δs (*OR* trig. *OR* coord.geom.)

A1 $\dfrac{PQ}{b} = \dfrac{\frac{1}{2}a}{\frac{1}{2}(b-a)} \Rightarrow PQ = \dfrac{ba}{b-a}$

M1 so a guard at a midpoint can see $4b - 2PQ$

A1 $= \dfrac{2b(2b-3a)}{b-a}$ (might be given as all but $\dfrac{2ba}{b-a}$ or as a fraction of the perimeter)

④

B1 Recognition that $b = 3a$ is the case when guard at *M* / *C* equally preferable (*P* at corner in the two *M* cases)

M1A1 Relevant algebra for comparison of one case $\dfrac{4b^2}{b+a} - \dfrac{b^2}{a} \equiv \dfrac{b^2}{a(b+a)}(3a-b)$

A1 Correct conclusion: Guard stands at *C* for $b < 3a$ and at *M* for $b > 3a$

M1A1 Relevant algebra $\dfrac{4b^2}{b+a} - \dfrac{2b(2b-3a)}{b-a} \equiv \dfrac{2ba}{(b+a)(b-a)}(3a-b)$

A1 Correct conclusion: Guard stands at *C* for $b < 3a$ and at *M* for $b > 3a$

⑦

Overall, I am anticipating that most attempts will do the Corner scenario and ***one*** of the Middle scenarios. This will allow for a maximum of 12 = 5 (for the Corner work) + 4 (for the Middle work) + 3 (for the comparison). In this circumstance, it won't generally be suitable to give the **B1** for the $b = 3a$ observation.

SI-2015/Q4

M1 When P is at $(x, \frac{1}{4}x^2)$ ... and makes an angle of θ with the positive x-axis

A1 ... the lower end, Q, is at $\left(x - b\cos\theta,\ \frac{1}{4}x^2 - b\sin\theta\right)$

M1 Also, $y = \frac{1}{4}x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x = \tan\theta$

A1 $\Rightarrow x = 2\tan\theta$ i.e. $P = \left(2\tan\theta,\ \tan^2\theta\right)$

A1A1 so that $Q = \left(2\tan\theta - b\cos\theta,\ \tan^2\theta - b\sin\theta\right)$ obtained *legitimately* (**Given Answer**)

⑥

M1A1 When $x = 0$, $2\tan\alpha = b\cos\alpha \Rightarrow b = \dfrac{2\tan\alpha}{\cos\alpha}$

M1A1 Substg. into y-coordinate $\Rightarrow$ $y_A = \tan^2\alpha - 2\tan\alpha\dfrac{\sin\alpha}{\cos\alpha} = -\tan^2\alpha$

④

M1A1 Eqn. of line AP is $y = x\tan\alpha - \tan^2\alpha$

M1A1 Area between curve and line is $\int\left(\frac{1}{4}x^2 - \left[x\tan\alpha - \tan^2\alpha\right]\right)\mathrm{d}x$

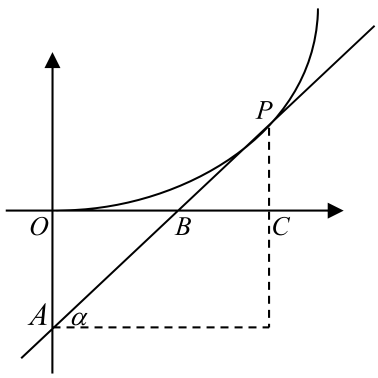
B1 Correct limits (0, 2tan α)

A1A1 $= \left[\frac{1}{12}x^3 - \frac{1}{2}x^2\tan\alpha + x\tan^2\alpha\right]$ (Any 2 correct terms; all 3)

A1A1 $= \frac{2}{3}\tan^3\alpha - 2\tan^3\alpha + 2\tan^3\alpha$ (Any 2 correct terms; all 3 **FT**)

A1 $= \frac{2}{3}\tan^3\alpha$ obtained *legitimately* (**Given Answer**) ⑩

ALTERNATIVE



M1 A1 for obtaining the "conversion factor" $b\cos\alpha = 2\tan\alpha$ **or** $\tan^2\alpha = \frac{1}{2}b\sin\alpha$

M1 A1 for distances $OB = BC$ $(= \frac{1}{2}b\cos\alpha)$ and so $PC = OA = \tan^2\alpha$

M1 A1 giving $\Delta OAB \equiv \Delta CPB$

A1 $\Rightarrow$ Area is $\int \frac{1}{4}x^2\ \mathrm{d}x$

B1 Correct limits (0, 2 $\tan\alpha$) used

A1 A1 Correct integration; correct **Given Answer**

ALTERNATIVE Translate whole thing up by $\tan^2\alpha$ and calculate $\int_0^{b\cos\alpha}\left(\frac{1}{4}x^2 + \tan^2\alpha\right)\mathrm{d}x - \Delta$

SI-2015/Q5

(i) **M1A1** $\text{f}(x) = \left[\dfrac{(t-1)^x}{x}\right]_1^3$

A1 $= \dfrac{2^x}{x}$ ③

M1 Differentiating by use of *Quotient RuleOR* taking logs.anddiffg. implicitly)

B1 for $\dfrac{\text{d}}{\text{d}x}\left(2^x\right) = 2^x.\ln 2$ seen at any stage

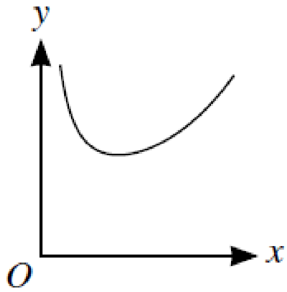
A1 $\dfrac{\text{d}y}{\text{d}x} = \dfrac{x.2^x.\ln 2 - 2^x}{x^2}$

A1 TP at $\left(\dfrac{1}{\ln 2},\ (\text{e}\ln 2)\right)$ (y-coordinate not required)

B1 Jusitfying that the TP is a minimum ⑤

G1 Generally correct ∪-shape

G1 Asymptotic to y-axis
and TP in **FT** correct position



②

(ii) **M1** Let $u^2 = 1 + x^2 - 2xt$

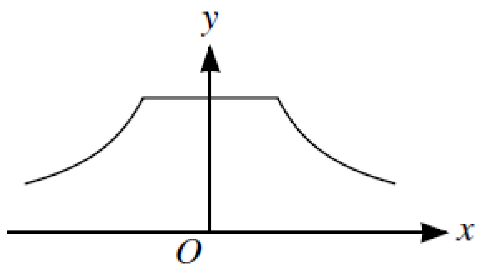
A1 $2u\ \text{d}u = -\ 2x\text{d}t$

B1 t: $(-1, 1) \rightarrow u$: $(\ |\ 1 + x\ |, |\ 1 - x\ |\ )$ Correct limits seen at any stage

M1A1 Full substn. attempt; correct $\text{g}(x) = \dfrac{-1}{x}\displaystyle\int 1.\text{d}u$

A1 $\text{g}(x) = \dfrac{1}{x}\left(|1+x| - |1-x|\right)$ ∫n. may be done directly, but be strict on the limits

or $\text{g}(x) = \begin{cases} -\dfrac{2}{x} & x < -1 \\ 2 & -1 \le x \le 1 \\ \dfrac{2}{x} & x > 1 \end{cases}$



(Must have completely correct three intervals: $x < -1$, $-1 \le x \le 1$, $x > 1$)

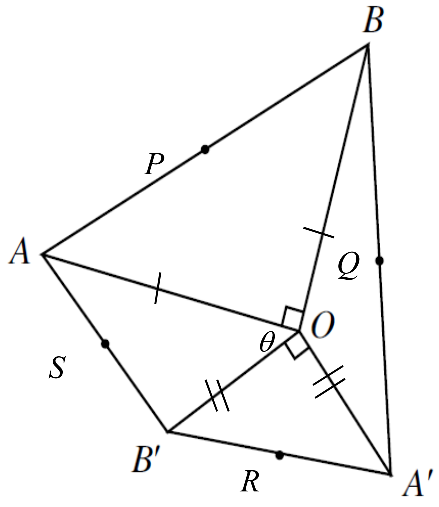
M1 Graph split into two or three regions

A1 A1 Reciprocal graphs on LHS & RHS (must be asymptotic to x-axis)
(Allow even if they approach y-axis also)

A1 Horizontal line for middle segment

⑩

SI-2015/Q6



Let P, Q, R and S be the midpoints of sides (as shown)
Then

M1A1 $\mathbf{p} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{q} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}'$,

$\mathbf{r} = \frac{1}{2}\mathbf{a}' + \frac{1}{2}\mathbf{b}'$, $\mathbf{s} = \frac{1}{2}\mathbf{b}' + \frac{1}{2}\mathbf{a}$

and

M1A1 $\overrightarrow{PQ} = \overrightarrow{SR} = \frac{1}{2}(\mathbf{a}' - \mathbf{a})$

A1 $\overrightarrow{QR} = \overrightarrow{PS} = \frac{1}{2}(\mathbf{b}' - \mathbf{b})$

A1 so that $PQSR$ is a //gm.
(opposite sides // and equal)

⑥

M1 $\overrightarrow{PQ} \bullet \overrightarrow{QR} = \overrightarrow{PQ} \bullet \overrightarrow{QS} = \frac{1}{2}(\mathbf{a}' - \mathbf{a}) \bullet \frac{1}{2}(\mathbf{b}' - \mathbf{b})$ for use of the scalar product

A1 $= \frac{1}{4}(\mathbf{a}' \bullet \mathbf{b}' - \mathbf{a} \bullet \mathbf{b}' - \mathbf{a}' \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b})$ Do not accept $\mathbf{a}'\mathbf{b}'$ etc.

M1 Use of perpendicularity of OA, OB and OA', OB'

$= -\frac{1}{4}(\mathbf{a} \bullet \mathbf{b}' + \mathbf{a}' \bullet \mathbf{b})$

M1 $\angle AOB' = \theta \Rightarrow \angle A'OB = 180^\circ - \theta$; and $\cos(180^\circ - \theta) = -\cos\theta$

A1 $= 0$ since $\mathbf{a} \bullet \mathbf{b}' = ab'\cos\theta$ and $\mathbf{a}' \bullet \mathbf{b} = -a'b\cos\theta$

and we are given that $a = b$ and $a' = b'$

A1 so that $PQRS$ is a rectangle (adjacent sides perpendicular)

⑥

B1 $PQ^2 = SR^2 = \overrightarrow{PQ} \bullet \overrightarrow{PQ} = \frac{1}{4}(a^2 + (a')^2 - 2\mathbf{a} \bullet \mathbf{a}')$

B1 $QR^2 = PS^2 = \frac{1}{4}(b^2 + (b')^2 - 2\mathbf{b} \bullet \mathbf{b}')$

M1 Since $a = b$, $a' = b'$ and $\mathbf{a} \bullet \mathbf{a}' = aa'\cos(90^\circ + \theta)$, $\mathbf{b} \bullet \mathbf{ab}' = bb'\cos(90^\circ + \theta)$

A1 it follows that $PQRS$ is a square (adjacent sides equal)

④

M1A1 Area $PQRS = \frac{1}{4}(a^2 + (a')^2 - 2aa'\cos[90^\circ + \theta])$

M1 ... which is maximal when $\cos[90^\circ + \theta] = -1$

A1 i.e. when $\theta = 90^\circ$

④

SI-2015/Q7

M1 $f'(x) = 6ax - 18x^2$

$= 6x(a - 3x)$

A1A1 $= 0$ for $x = 0$ and $x = \frac{1}{3}a$

A1A1 $f(0) = 0$ $f(\frac{1}{3}a) = \frac{1}{9}a^3$

A1 (Min. TP) (Max. TP) since $f(x)$ is a 'negative' cubic

(f(0) = 0 and the TPs may be shown on a sketch – award the marks here if necessary)

⑥

M1 Evaluating at the endpoints

A1A1 $f(-\frac{1}{3}) = \frac{1}{9}(3a + 2)$; $f(1) = 3a - 6$

③

M1 $\frac{1}{9}(3a + 2) \geq \frac{1}{9}a^3 \Leftrightarrow a^3 - 3a - 2 \leq 0$

M1 $\Leftrightarrow (a + 1)^2(a - 2) \leq 0$

A1 and since $a \geq 0$, $a \leq 2$

M1 $\frac{1}{9}a^3 \geq 3a - 6 \Leftrightarrow a^3 - 27a + 54 \geq 0$

M1 $\Leftrightarrow (a - 3)^2(a + 6) \geq 0$

A1 which holds for all $a \geq 0$

M1 $\frac{1}{9}(3a + 2) \geq 3a - 6 \Leftrightarrow 3a + 2 \geq 27a - 54$

$\Leftrightarrow 8(3a - 7) \leq 0$

A1 $\Leftrightarrow a \leq \frac{7}{3}$ (which, actually, affects nothing, but working should appear)

⑧

Thus

B1B1B1 $$M(a) = \begin{cases} \frac{1}{9}(3a+2) & 0 \leq a \leq 2 \\ \frac{1}{9}a^3 & 2 \leq a \leq 3 \\ 3a-6 & a \geq 3 \end{cases}$$ (Ignore 'non-unique' allocation of endpoints) ③

(Do not award marks for correct answers unsupported or from incorrect working)

SI-2015/Q8

(i) $S = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$

M1 $S = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$ Method

M1 $2S = n \times (n+1)$ Adding

A1 $S = \frac{1}{2}n(n+1)$ obtained *legitimately* (**Given Answer**)

(Allow alternatives using induction or the *Method of Differences*, for instance, but **NOT** by stating that it is an AP and just quoting a formula; ditto Δ-number formula)

③

(ii) $(N-m)^k + m^k$ (k odd)

M1A1 $= N^k - \binom{k}{1}mN^{k-1} + \binom{k}{2}m^2N^{k-2} - \ldots + \binom{k}{k-1}m^{k-1}N - m^k + m^k$

E1 which is clearly divisible by N (since each term has a factor of N)

(Allow alternatives using induction, for instance)

③

Let $S = 1^k + 2^k + \ldots + n^k$ an odd no. of terms

M1 $= 0^k + 1^k + 2^k + \ldots + n^k$ an even no. of terms

M1 $= \left[(n-0)^k + 0^k\right] + \left[(n-1)^k + 1^k\right] + \ldots + \left[(\tfrac{1}{2}n + \tfrac{1}{2})^k + (\tfrac{1}{2}n - \tfrac{1}{2})^k\right]$

(no need to demonstrate final pairing but must explain fully the pairing up *or* the single extra n^k term)

E1 and, by (ii), each term is divisible by n.

③

For $S = 1^k + 2^k + \ldots + n^k$ an even no. of terms

M1 $= 0^k + 1^k + 2^k + \ldots + n^k$ an odd no. of terms

M1 $= \left[(n-0)^k + 0^k\right] + \left[(n-1)^k + 1^k\right] + \ldots + \left[(\tfrac{1}{2}n + 1)^k + (\tfrac{1}{2}n - 1)^k\right] + \left(\tfrac{1}{2}n\right)^k$

(no need to demonstrate final pairing but must explain the pairing and note the separate, single term)

and, by (ii), each paired term is divisible by n

E1 and the final single term is divisible by $\frac{1}{2}n \Rightarrow$ required result

③

M1 By the above result … for n even, so that $(n+1)$ is odd

A1 $(n+1) \mid 1^k + 2^k + \ldots + n^k + (n+1)^k$

E1 $(n+1) \mid S + (n+1)^k \Rightarrow (n+1) \mid S$

M1 By the above result … for n odd, so that $(n+1)$ is even

A1 $\frac{1}{2}(n+1) \mid 1^k + 2^k + \ldots + n^k + (n+1)^k$

E1 $\frac{1}{2}(n+1) \mid S + (n+1)^k \Rightarrow \frac{1}{2}(n+1) \mid S$ (as $\frac{1}{2}(n+1)$ is an integer)

E1 Since $\text{hcf}(n, n+1) = 1 \Rightarrow \text{hcf}(\frac{1}{2}n, n+1) = 1$ for n even

E1 and $\text{hcf}(n, \frac{1}{2}(n+1)) = 1$ for n odd

So it follows that $\frac{1}{2}n(n+1) \mid S$ for all positive integers n ⑧

SI/15/Q9

M1	Time taken to land (at the level of the projection) (from $y = ut\sin\alpha - \frac{1}{2}gt^2$, $y = 0, t \neq 0$)
A1	is $t = \dfrac{2u\sin\alpha}{g}$ (may be implicit)
M1	Bullet fired at time $t\left(0 \le t \le \dfrac{\pi}{6\lambda}\right)$ lands at time
A1	$T_L = t + \dfrac{2u}{g}\sin\left(\dfrac{\pi}{3} - \lambda t\right)$
M1A1	$\dfrac{dT_L}{dt} = 1 - \dfrac{2\lambda u}{g}\cos\left(\dfrac{\pi}{3} - \lambda t\right) = \dfrac{1}{k}\left\{k - \cos\left(\dfrac{\pi}{3} - \lambda t\right)\right\}$
A1	$= 0$ when $k = \cos\left(\dfrac{\pi}{3} - \lambda t\right)$
M1A1	Horizontal range is $R = \dfrac{2u^2\sin\alpha\cos\alpha}{g}$ (from $y = ut\sin\alpha - \frac{1}{2}gt^2$ with above time)
A1	$\Rightarrow R_L = \dfrac{2u^2}{g}k\sqrt{1-k^2}$ obtained *legitimately* (**Given Answer**) ⑩
M1A1	$\dfrac{d^2T_L}{dt^2} = -\dfrac{2\lambda^2 u}{g}\sin\left(\dfrac{\pi}{3} - \lambda t\right) < 0 \Rightarrow$ maximum distance
M1A1	$0 \le t \le \dfrac{\pi}{6\lambda}$ in $k = \cos\left(\dfrac{\pi}{3} - \lambda t\right) \Rightarrow \dfrac{1}{2} \le k \le \dfrac{\sqrt{3}}{2}$ ④
M1	If $k < \frac{1}{2}$ then $\dfrac{dT_L}{dt} < 0$ throughout the gun's firing …
A1	… and T_L is a (strictly) decreasing function.
M1	Then T_L max. occurs at $t = 0$
A1	i.e. $\alpha = \dfrac{\pi}{3}$
M1A1	and $R_L = \dfrac{2u^2}{g} \times \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{u^2\sqrt{3}}{2g}$ ⑥

SI-2015/Q10

B1	Speed of rain relative to bus is $v\cos\theta - u$ (or $u - v\cos\theta$ if negative)
M1A1	When $u = 0$, $A \propto hv\cos\theta + av\sin\theta$ (width of bus and time units may be included as factors)
E1	When $v\cos\theta - u > 0$, rain hitting top of bus is the same, and rain hits back of bus as before, but with $v\cos\theta - u$ instead of $v\cos\theta$
E1	When $v\cos\theta - u < 0$, rain hitting top of bus is the same, and rain hits front of bus as before, but with $u - v\cos\theta$ instead of $v\cos\theta$
A1	Together, $A \propto h\,\lvert v\cos\theta - u\rvert + av\sin\theta$ Fully justified (**Given Answer**)

⑥

M1	Journey time $\propto \frac{1}{u}$ so we need to minimise
A1	$J = \frac{av\sin\theta}{u} + \frac{h\,\lvert v\cos\theta - u\rvert}{u}$ (Ignore additional constant-of-proportionality factors)
M1	For $v\cos\theta - u > 0$,
	if $w \le v\cos\theta$, we minimise $J = \frac{av\sin\theta}{u} + \frac{hv\cos\theta}{u} - h$
E1	and this decreases as u increases
E1	and this is done by choosing u as large as possible; i.e. $u = w$
M1	For $u - v\cos\theta > 0$,
	we minimise $J = \frac{av\sin\theta}{u} - \frac{hv\cos\theta}{u} + h$
E1	and this decreases as u increases if $a\sin\theta > h\cos\theta$
E1	so we again choose u as large as possible; i.e. $u = w$
	[Note: minimisation may be justified by calculus in either case or both.]

⑧

M1	If $a\sin\theta < h\cos\theta$, then J increases with u when u exceeds $v\cos\theta$
A1	so we choose $u = v\cos\theta$ in this case

②

M1A1	If $a\sin\theta = h\cos\theta$ then J is independent of u, so we may as well take $u = w$ ②

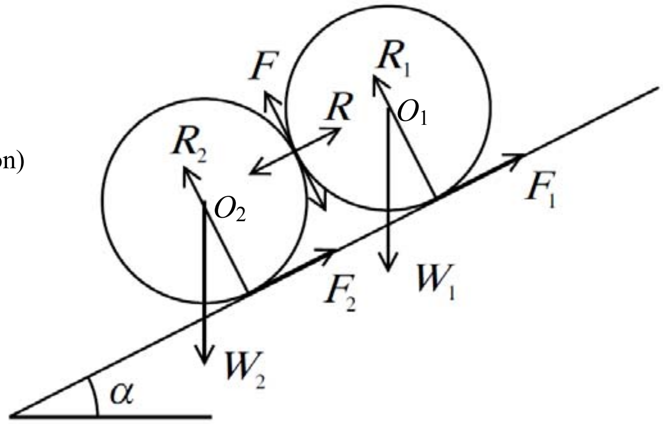
M1	Replacing θ by $180^\circ - \theta$ gives $J = \frac{av\sin\theta}{u} + \frac{hv\cos\theta}{u} + h$
A1	Which always decreases as u increases, so take $u = w$ again

②

SI-2015/Q11

(i) **B1** ↺O_1: $F = F_1$

↺O_2: $F = F_2$ (Both, with reason)



①

(ii) **B1** Res^g. ||plane (for C_1): $F_1 + R = W_1 \sin\alpha$ ①

B1 Res^g.⊥^r. plane (for C_1): $R_1 + F = W_1 \cos\alpha$ ②

B1 Res^g.||plane (for C_2): $F_2 - R = W_2 \sin\alpha$ ③

B1 Res^g.⊥^r. plane (for C_2): $R_2 - F = W_2 \cos\alpha$ ④

Max 4 marks to be given for four independent statements (though only 3 are required).
One or other of

Res^g.||plane (for system) : $F_1 + F_2 = (W_1 + W_2)\sin\alpha$

Res^g.⊥^r. plane (for system) : $R_1 + R_2 = (W_1 + W_2)\cos\alpha$

may also appear instead of one or more of the above.

(F_1 and F_2 may or may not appear in these statements as F, but should do so below) ④

M1A1 Equating for $\sin\alpha$: $\frac{F+R}{W_1} = \frac{F-R}{W_2}$ using ① and ③

M1A1 Re-arranging for F in terms of R: $F = \left(\frac{W_1 + W_2}{W_1 - W_2}\right)R$

M1 Use of the Friction Law, $F \le \mu R$

A1 $\Rightarrow \frac{W_1 + W_2}{W_1 - W_2} \le \mu$ obtained *legitimately* (**Given Answer**) ⑥

M1A1 (e.g.) ①÷② $\Rightarrow$ $\tan\alpha = \dfrac{F+R}{R_1+F}$

M1A1 Subst[g]. for R $= \dfrac{F + F\left(\dfrac{W_1 - W_2}{W_1 + W_2}\right)}{R_1 + F}$ using $R = \left(\dfrac{W_1 - W_2}{W_1 + W_2}\right)F$

A1 $= \dfrac{F\left(\dfrac{2W_1}{W_1 + W_2}\right)}{R_1 + F_1}$

M1A1 Subst[g]. for R_1 (correct inequality) using Friction Law $F_1 \le \mu_1 R_1 \Leftrightarrow R_1 \ge \dfrac{F_1}{\mu_1}$

$$\le \frac{F\left(\dfrac{2W_1}{W_1 + W_2}\right)}{\dfrac{F_1}{\mu_1} + F_1}$$

M1 Tidying-up algebra $= \dfrac{F\left(\dfrac{2W_1}{W_1 + W_2}\right)}{F\left(\dfrac{1+\mu_1}{\mu_1}\right)}$

A1 $\Rightarrow$ $\tan\alpha \le \dfrac{2\mu_1 W_1}{(1+\mu_1)(W_1 + W_2)}$ obtained *legitimately* (**Given Answer)**

⑨

SI-2015/Q12

(i) **M1A1** P(exactly r out of n need surgery) = $\binom{n}{r}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}$ (A binomial prob. term; correct)

②

(ii) **M1** $P(S=r)=\sum_{n=r}^{\infty}\frac{e^{-8}8^n}{n!}\times\frac{n!}{r!(n-r)!}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}$ Attempt at sum of appropriate product terms

B1B1A1 Limits✓✓ All internal terms correct; allow ^nC_r for the A mark

M1 $=\frac{e^{-8}}{r!}\sum_{n=r}^{\infty}\frac{8^n}{(n-r)!}\times\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}$ Factoring out these two terms

M1 $=\frac{e^{-8}}{r!}\sum_{n=r}^{\infty}\frac{8^n}{(n-r)!}\times\frac{3^{n-r}}{4^n}$ Attempting to deal with the powers of 3 and 4

A1 $=\frac{e^{-8}}{r!}\sum_{n=r}^{\infty}\frac{2^n\times 3^{n-r}}{(n-r)!}$ Correctly

M1 $=\frac{e^{-8}\times 2^r}{r!}\sum_{n=r}^{\infty}\frac{6^{n-r}}{(n-r)!}$ Splitting off the extra powers of 2 ready to …

M1 $=\frac{e^{-8}\times 2^r}{r!}\sum_{m=0}^{\infty}\frac{6^m}{m!}$ … adjust the lower limit (i.e. using $m=n-r$)

A1 $=\frac{e^{-8}\times 2^r}{r!}\times e^6$ i.e. $\frac{e^{-2}\times 2^r}{r!}$

A1 … which is Poisson with mean 2 (Give **B1** for noting this without the working) ⑪

(iii) **M1** $P(M=8\,|M+T=12)$ Identifying correct conditional probability outcome

A1A1A1 $=\frac{\frac{e^{-2}\times 2^8}{8!}\times\frac{e^{-2}\times 2^4}{4!}}{\frac{e^{-4}\times 4^{12}}{12!}}$ One A mark for each correct term (& no extras for 3rd A mark)

A1A1 $=\frac{2^{12}\times 12!}{4^{12}\times 8!\times 4!}$ Powers of e cancelled; factorials in correct part of the fraction – (unsimplified is okay at this stage)

A1 $=\frac{495}{4096}$ ⑦

SI-2015/Q13

Reminder

A : the 1st 6 arises on the n^{th} throw
B : at least one 5 arises before the 1st 6
C : at least one 4 arises before the 1st 6
D : exactly one 5 arises before the 1st 6
E : exactly one 4 arises before the 1st 6

(i) **M1A1** $\mathrm{P}(A) = \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$ ②

(ii) **M1A1** By symmetry (either a 5 or a 6 arises before the other), $\mathrm{P}(B) = \frac{1}{2}$ ②

(iii) **M1** The first 4s, 5s, 6s can arise in the orders **<u>456</u>**, 465, **<u>546</u>**, 564, 645, 654

A1 $\Rightarrow \mathrm{P}(B \cap C) = \frac{1}{3}$ (i.e. "by symmetry" but with three pairs)

②

(iv) **M1A1A1** $\mathrm{P}(D) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right)\left(\frac{1}{6}\right) + \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right)^2\left(\frac{1}{6}\right) + \ldots$

M1 for infinite series with 1st term ✓; A1 for 2nd term ✓; A1 for 3rd term and following pattern ✓

M1 $= \left(\frac{1}{36}\right)\left\{1 + 2\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \ldots\right\}$ For factorisation and an infinite series

M1 $= \left(\frac{1}{36}\right)\left(1 - \frac{2}{3}\right)^{-2}$ Use of the given series result

A1 $= \frac{1}{4}$

⑥

(v) **M1** $\mathrm{P}(D \cup E) = \mathrm{P}(D) + \mathrm{P}(E) - \mathrm{P}(D \cap E)$ Stated or used

B1 $\mathrm{P}(E) = \mathrm{P}(D) =$ answer to (iv) Stated or used anywhere

M1A1A1 $\mathrm{P}(D \cap E) = \left(\frac{2}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \binom{3}{1}\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \binom{4}{2}\left(\frac{3}{6}\right)^2\left(\frac{2}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \ldots$

M1 for infinite series with 1st term ✓; A1 for 2nd term ✓; A1 for 3rd term and following pattern ✓

M1 $= \left(\frac{1}{108}\right)\left\{1 + 3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + \ldots\right\}$ For factorisation and an infinite series

M1 $= \left(\frac{1}{108}\right)\left(1 - \frac{1}{2}\right)^{-3}$ Use of the given series result

A1 $\Rightarrow \mathrm{P}(D \cup E) = \frac{1}{2} - \frac{2}{27} = \frac{23}{54}$

⑧

Question 1

(i)	$\frac{d}{dx}(x - \ln(1+x)) = 1 - \frac{1}{1+x}$	**B1**
	For $x > 0$, $\frac{1}{1+x} < 1$	**M1**
	Therefore $\frac{d}{dx}(x - \ln(1+x)) > 0$ for $x > 0$	**A1**
	If $x = 0$, $x - \ln(1+x) = 0$	
	Therefore $x - \ln(1+x)$ is positive for all positive x.	**B1**
	Therefore $\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) > 0$ for all positive k.	
	So, $\sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right)$	**B1**
	$\ln\left(1 + \frac{1}{k}\right) = \ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln k$	**M1**
	So, $\sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{n} \ln(k+1) - \ln k = \ln(n+1) - \ln 1$	**M1**
	Therefore, $\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$	**A1**
(ii)	$\frac{d}{dx}(x + \ln(1-x)) = 1 - \frac{1}{1-x}$	**B1**
	For $0 < x < 1$, $\frac{1}{1-x} > 1$	**M1**
	Therefore $\frac{d}{dx}(x + \ln(1-x)) < 0$ for $0 < x < 1$.	**A1**
	If $x = 0$, $x + \ln(1-x) = 0$	
	Therefore $x + \ln(1-x)$ is negative for $0 < x < 1$.	**B1**
	Therefore $\frac{1}{k^2} + \ln\left(1 - \frac{1}{k^2}\right) < 0$ for all $k > 1$.	
	So, $\sum_{k=2}^{n} \frac{1}{k^2} < \sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right)$	**B1**
	$-\ln\left(1 - \frac{1}{k^2}\right) = -\ln\left(\frac{k^2-1}{k^2}\right) = -\ln(k-1) + 2\ln k - \ln(k+1)$	**M1 M1 A1**
	So, $\sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right) = \ln 2 + \ln n - \ln(n+1)$	**M1 A1**
	As $n \to \infty$, $\ln n - \ln(n+1) \to 0$	**B1**
	Therefore, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \sum_{k=2}^{\infty} \frac{1}{k^2} < 1 + \ln 2$	**A1**

Question 1

Note that the statement of the question requires the use of a particular method in both parts.

(i)	
B1	Correct differentiation of the expression.
M1	Consideration of the sign of the derivative for positive values of x.
A1	Deduction that the derivative is positive for all positive values of x.
B1	Clear explanation that $x - \ln(1+x)$ is positive for all positive x. ***Note that answer is given in the question.***
B1	Use of $x = \frac{1}{k}$ and summation.
M1	Manipulation of logarithmic expression to form difference.
M1	Attempt to simplify the sum (some pairs cancelled out within sum).
A1	Clear explanation of result. ***Note that answer is given in the question.***
(ii)	
B1	Correct differentiation of the expression.
M1	Consideration of the sign of the derivative for $0 < x < 1$.
A1	Deduction that the derivative is negative for this range of values.
B1	Deduction that $x + \ln(1-x)$ is negative for this range of values.
B1	Use of $x = \frac{1}{k^2}$ and summation.
M1	Expression within logarithm as a single fraction and numerator simplified.
M1	Logarithm split to change at least one product to a sum of logarithms or one quotient as a difference of logarithms.
A1	Complete split of logarithm to required form.
M1	Use of differences to simplify sum.
A1	$\ln 2$ correct.
B1	Correctly dealing with limit as $n \to \infty$. **Note that answers which use ∞ as the upper limit on the sum from the beginning must have clear justification of the limit. Those beginning with n as the upper limit must have** $\ln n - \ln(n+1)$ correct in simplified sum.
A1	Inclusion of $k = 1$ to the sum to reach the final answer. ***Note that answer is given in the question.***

Question 2

	$\angle ACB = \pi - 3\alpha$	**B1**
	$\dfrac{AB}{\sin(\pi - 3\alpha)} = \dfrac{x}{\sin\alpha}$	**M1 A1**
	$\sin(\pi - 3\alpha) = \sin 3\alpha$	**B1**
	$\therefore AB = \dfrac{x\sin 3\alpha}{\sin\alpha}$	
	$= \dfrac{x(\sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha)}{\sin\alpha}$	**M1**
	$= \dfrac{x(\sin\alpha \times (1 - 2\sin^2\alpha) + \cos\alpha \times 2\sin\alpha\cos\alpha)}{\sin\alpha}$	**M1 M1**
	$AB = (3 - 4\sin^2\alpha)x$	**A1**
	$DE = AB - BE - AD$ (or $DE = DB - BE$)	**B1**
	$DE = AB - BE - \frac{1}{2}AB = \frac{1}{2}AB - BE$	
	$DE = \dfrac{x}{2}(3 - 4\sin^2\alpha) - x\cos 2\alpha$	**B1 B1**
	$DE = \dfrac{x}{2}\left((3 - 4\sin^2\alpha) - 2(1 - 2\sin^2\alpha)\right) = \dfrac{x}{2}$	**M1 M1 A1**
	$\sin(\angle FCE) = \frac{DE}{x} = \frac{1}{2}$, so $\angle FCE = \frac{\pi}{6}$	**B1 M1 A1**
	Therefore $\angle ACF = \pi - 3\alpha - \left(\frac{\pi}{2} - 2\alpha\right) - \frac{\pi}{6} = \frac{\pi}{3} - \alpha$	**M1 M1**
	$\angle ACF = \dfrac{1}{3}(\pi - 3\alpha) = \dfrac{1}{3}\angle ACB$ So FC trisects the angle ACB	**A1**

Question 2

B1	Expression for $\angle ACB$ (may be implied by later working).
M1	Application of the sine rule.
A1	Correct statement.
B1	Does not need to be stated as long as implied in working.
M1	Use of $\sin(A + B)$ formula.
M1	Use of double angle formula for sin.
M1	Use of double angle formula for cos.
A1	Simplification of expression. ***Note that answer is given in the question.***
B1	Identification of this relationship between distances. (just $BD - BE$ is sufficient)
B1	Correct expression substituted for the length of BD.
B1	Correct expression substituted for the length of BE.
M1	Use of double angle formula for cos.
M1	Simplification of expression obtained.
A1	Correct expression independent of x.
B1	Identification of a right angled triangle to calculate $\sin(\angle FCE)$.
M1	Deduction that one of the lengths in sine of this angle is equal to DE.
A1	Value of the angle (Degrees or radians are both acceptable).
M1	Obtaining $\angle BCE = \frac{\pi}{2} - 2\alpha$
M1	Use of $\angle ACF = \angle ACB - \angle BCE - \angle FCE$.
A1	Expression to show that $\angle ACF = \frac{1}{3}\angle ACB$ and conclusion stated.

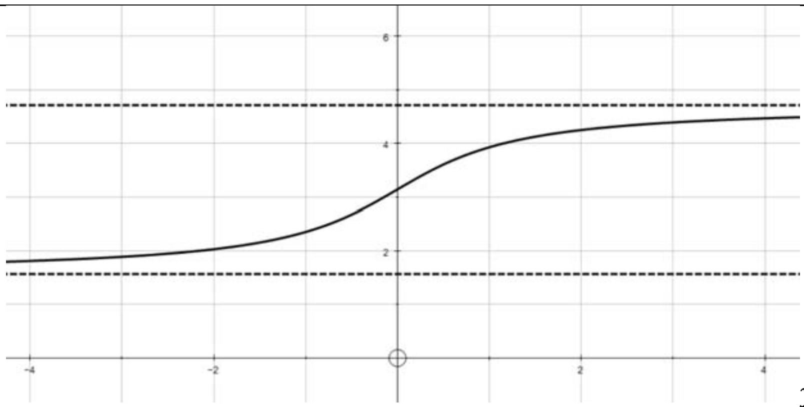
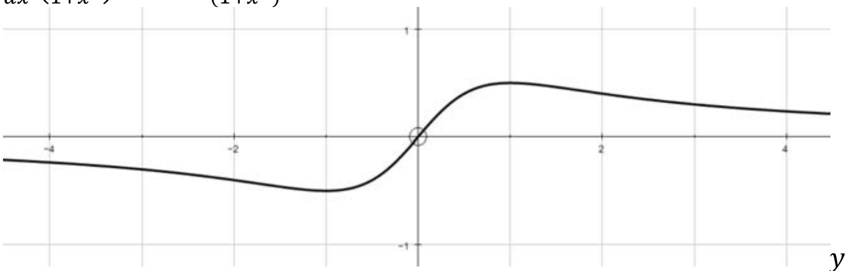
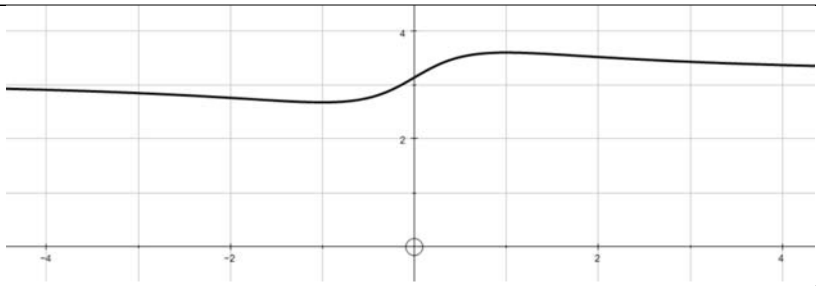
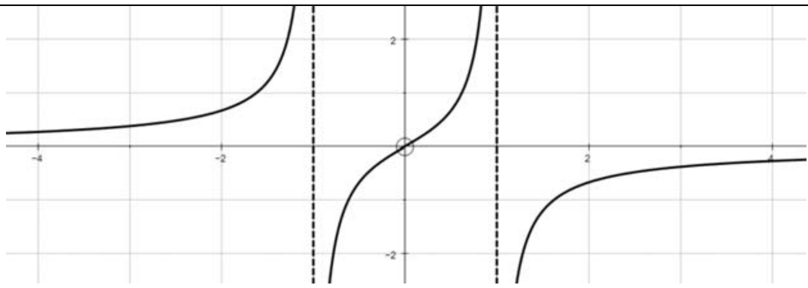
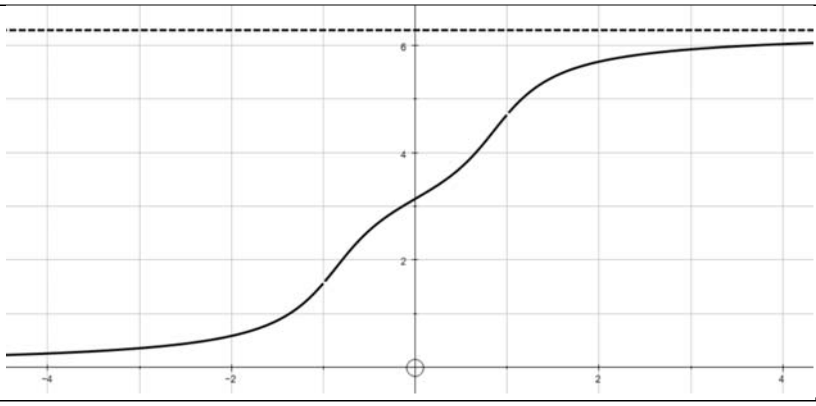
Question 3

	$T_8 - T_7$ is the number of triangles that can be made using a rod of length 8 and two other, shorter rods.	**M1**
	If the middle length rod has length 7 then the other rod can be 1, 2, 3, 4, 5 or 6.	**M1**
	If the middle length rod has length 6 then the other rod can be 2, 3, 4 or 5.	
	If the middle length rod has length 5 then the other rod can be 3 or 4.	**M1**
	$T_8 - T_7 = 2 + 4 + 6.$	**A1**
	Assume that the longest of the three rods has length 7:	**M1**
	If the middle length rod has length 6 then the other rod can be 1, 2, 3, 4 or 5.	**M1**
	If the middle length rod has length 5 then the other rod can be 2, 3 or 4.	
	If the middle length rod has length 4 then the other rod must be 3.	**M1**
	Therefore $T_7 - T_6 = 1 + 3 + 5.$	**A1**
	$T_8 - T_6 = T_8 - T_7 + T_7 - T_6 = 1 + 2 + 3 + 4 + 5 + 6.$	**A1**
	$T_{2m} - T_{2m-1} = 2 + 4 + \cdots + 2(m-1)$	**B1**
	$T_{2m} - T_{2m-2} = 1 + 2 + 3 + \cdots + 2(m-1)$	**B1**
	$T_4 = 3$. (The possibilities are $\{1,2,3\}, \{1,3,4\}$ and $\{2,3,4\}$.)	**B1**
	Substituting $m = 2$ into the equation gives $T_4 = \frac{1}{6}(2)(2-1)(4 \times 2 + 1) = 3.$	
	Therefore the formula is correct in the case $m = 2$.	**B1**
	Assume that the formula is correct in the case $m = k$: $T_{2(k+1)} = T_{2k} + \sum_{r=1}^{2k} r$	**M1**
	$T_{2(k+1)} = \frac{1}{6}k(k-1)(4k+1) + \frac{2k}{2}(2k+1)$	**M1**
	$T_{2(k+1)} = \frac{k}{6}[4k^2 - 3k - 1 + 12k + 6] = \frac{(k+1)}{6}(k)(4(k+1)+1)$, which is a statement of the formula where $m = k + 1$.	**M1**
	Therefore, by induction, $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$	**A1**
	$T_{2m} - T_{2m-1} = 2 + 4 + \cdots + 2(m-1) = m(m-1).$	**M1 A1**
	Therefore $T_{2m-1} = \frac{1}{6}m(m-1)(4m+1) - m(m-1).$	
	$T_{2m-1} = \frac{1}{6}m(m-1)(4m-5)$. (Or $T_{2m+1} = \frac{1}{6}m(m+1)(4m-1)$)	**A1**

Question 3

M1	Appreciation of the meaning of $T_8 - T_7$.
M1	Identify the number of possibilities for the length of the third rod in one case.
M1	Identify the set of possible cases and find numbers of possibilities for each.
A1	Clear explanation of the result. ***Note that answer is given in the question.***
M1	An attempt to work out $T_7 - T_6$.
M1	Correct calculation for any one defined case.
M1	Identification of a complete set of cases.
A1	Correct value for $T_7 - T_6$.
A1	Correct deduction of expression for $T_8 - T_6$.
B1	Correct expression. No justification is needed for this mark.
B1	Correct expression. No justification is needed for this mark.
B1	Correct justification that $T_4 = 3$. Requires sight of possibilities or other justification.
B1	Evidence of checking a base case. (Accept confirmation that $m = 1$ gives $T_2 = 0$ here.)
M1	Application of the previously deduced result.
M1	Substitution of formula for $m = k$ and the formula for the sum.
M1	Taking common factor to give a single product.
A1	Re-arrangement to show that it is a statement of the required formula when $m = k + 1$ and conclusion stated.
M1	Use of result from start of question.
A1	Correct summation of $2 + 4 + \cdots + 2(m - 1)$.
A1	Correct formula reached (any equivalent expression is acceptable).

Question 4

(i)	 $y = f(x)$	**B1** **B1**
(ii)	$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{(1+x^2)(1)-(x)(2x)}{(1+x^2)^2}$, so stationary points when $x = \pm 1$.  $y = \frac{x}{1+x^2}$	**B1** **B1** **M1** **A1** **A1**
	 $y = g(x)$	**B1** **B1** **B1** **B1**
(iii)	 $y = \frac{x}{1-x^2}$	**B1** **B1** **B1** **B1**
	 $y = h(x)$	**B1** **B1** **B1** **B1** **B1**

Question 4

Penalise additional sections to graphs (vertical translations by $\pm\pi$) only on the first occasion providing that the correct section is present in later parts.

B1	Correct shape.
B1	Asymptotes $y = \frac{\pi}{2}$ and $y = \frac{3\pi}{2}$ shown.
B1	Rotational symmetry about the point $(0,0)$.
B1	Correct shape.
M1	Differentiation to find stationary points.
A1	Correct stationary points - $\left(\pm 1, \pm\frac{1}{2}\right)$. ($x$-coordinates)
A1	Correct y-coordinates.
B1	Rotational symmetry about the point $(0, \pi)$.
B1	Correct shape.
B1	Stationary points have same x-coordinates as previous graph. (Follow through incorrect stationary points in previous graph if consistent here).
B1	Correct co-ordinates for stationary points - $\left(\pm 1, \pi \pm \arctan\frac{1}{2}\right)$
B1	Correct asymptotes $x = \pm 1$.
B1	x-axis as an asymptote.
B1	Middle section correct shape.
B1	Outside sections correct shape.
B1	Section for $-1 < x < 1$ correct shape.
B1	$h(-1) = \frac{\pi}{2}$.
B1	$h(1) = \frac{3\pi}{2}$.
B1	Section for $x > 1$ correct with asymptote $y = 2\pi$.
B1	Section for $x > -1$ correct with asymptote $y = 0$ or a rotation of $x > 1$ section about $(0, \pi)$.

Question 5

(i)	$\tan S_1 = \tan\left(\arctan\frac{1}{2}\right) = \frac{1}{1+1}$, so the formula is correct for $n = 1$.	**B1**
	Assume that $\tan S_k = \frac{k}{k+1}$:	
	$S_{k+1} = S_k + \arctan\frac{1}{2(k+1)^2}$.	**M1**
	$\tan S_{k+1} = \frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}$	**M1**
	$\tan S_{k+1} = \frac{2k(k+1)^2+(k+1)}{2(k+1)^3-k}$, which simplifies to $\tan S_{k+1} = \frac{(k+1)}{(k+1)+1}$.	**M1 A1**
	Hence, by induction $\tan S_n = \frac{n}{n+1}$.	**A1**
	Clearly, $S_1 = \arctan\left(\frac{1}{2}\right)$.	**B1**
	Suppose that it is not true that $S_n = \arctan\left(\frac{n}{n+1}\right)$ for all values of n. Then there is a smallest positive value, k such that $S_k \neq \arctan\left(\frac{k}{k+1}\right)$.	
	Since $S_k > S_{k-1}$, $S_{k-1} = \arctan\left(\frac{k-1}{k}\right)$ and $\tan S_k = \frac{k}{k+1}$, but $S_k \neq \arctan\left(\frac{k}{k+1}\right)$ $S_k - S_{k-1} > \pi$.	**M1 M1**
	However, $S_k - S_{k-1} = \arctan\left(\frac{1}{2k^2}\right) < \frac{\pi}{2}$, so this is not possible.	**A1**
	Therefore $S_n = \arctan\left(\frac{n}{n+1}\right)$.	**A1**
(ii)	$\tan 2\alpha_n = \frac{4n^2}{4n^4-1}$.	**M1 A1**
	So, $\frac{2\tan\alpha_n}{1-\tan^2\alpha_n} = \frac{4n^2}{4n^4-1}$	**B1**
	Which simplifies to $2n^2\tan^2\alpha_n - (1-4n^2)\tan\alpha_n - 2n^2 = 0$	**M1 A1**
	$(\tan\alpha_n + 2n^2)(2n^2\tan\alpha_n - 1) = 0$	**A1**
	Since α_n must be acute, $\tan\alpha_n$ cannot equal $-2n^2$.	**B1**
	Therefore $\alpha_n = \arctan\left(\frac{1}{2n^2}\right)$.	
	$\sum_{n=1}^{\infty}\alpha_n = \lim_{n\to\infty} S_n = \arctan 1 = \frac{\pi}{4}$.	**M1 A1**

Question 5

B1	Confirmation that the formula is correct for $n = 1$.
M1	Expression of S_{k+1} in terms of S_k.
M1	Use of $\tan(A + B)$ formula.
M1	Simplification of fraction.
A1	Expression of S_{k+1} to show that it matches result.
A1	Conclusion stated.
B1	Confirmation for $n = 1$.
M1	Observation that $S_k - S_{k-1} > 0$
M1	Evidence of understanding that successive values of x with the same value of $\tan x$ must differ by π.
A1	Evidence of understanding that $S_k - S_{k-1}$ cannot be sufficiently large for S_k to be of the form $\arctan x$ if S_{k-1} is.
A1	Clear justification.
M1	Identification of the relevant sides of the triangle (diagram is sufficient).
A1	Correct expression for $\tan 2\alpha_n$.
B1	Use of double angle formula.
M1	Rearrangement to remove fractions.
A1	Correct quadratic reached.
A1	Quadratic factorised.
B1	Irrelevant case eliminated (must be justified).
M1	Sum expressed as limit of S_n
A1	Correct value justified. ***Note that answer is given in the question.***

Question 6

(i)	$\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\dfrac{1}{\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$	**B1**
	$\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\dfrac{1}{\left(\cos\frac{\pi}{4}\cos\frac{x}{2}+\sin\frac{\pi}{4}\sin\frac{x}{2}\right)^2}$	**B1**
	$\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\dfrac{2}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}$	
	$\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2\equiv\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\sin^2\frac{x}{2}=1+\sin x$	**M1**
	Therefore, $\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{2}{1+\sin x}$	**M1 A1**
	Therefore, $\int\frac{1}{1+\sin x}\,dx=-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c$.	**M1 A1**
(ii)	Limits: $x=\pi\rightarrow y=0$ $x=0\rightarrow y=\pi$	
	$\frac{dy}{dx}=-1$	**B1**
	$\sin(\pi-x)=\sin x$	**B1**
	Therefore, $\int_0^\pi xf(\sin x)\,dx=\int_\pi^0(\pi-y)\,f(\sin(\pi-y))(-1)dy$	
	$\int_0^\pi xf(\sin x)\,dx=\int_0^\pi(\pi-x)\,f(\sin x)dx$	
	So, $2\int_0^\pi xf(\sin x)\,dx=\pi\int_0^\pi f(\sin x)dx$	**M1**
	$\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx$	**A1**
	$\int_0^\pi\frac{x}{1+\sin x}\,dx=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\sin x}\,dx$, and applying the result from part (i):	
	$\int_0^\pi\frac{1}{1+\sin x}\,dx=\left[-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]_0^\pi=\left(-\tan\left(-\frac{\pi}{4}\right)\right)-\left(-\tan\left(\frac{\pi}{4}\right)\right)=2.$	**B1**
	$\int_0^\pi\frac{x}{1+\sin x}\,dx=\frac{\pi}{2}(2)=\pi$	**B1**
(iii)	Consider $\int_0^\pi x^3f(\sin x)\,dx$. Making the substitution $y=\pi-x$:	
	$\int_0^\pi x^3f(\sin x)\,dx=\int_\pi^0(\pi-y)^3\,f(\sin(\pi-y))(-1)dy$	**M1 A1**
	So, $\int_0^\pi x^3f(\sin x)\,dx=\int_0^\pi(\pi-x)^3\,f(\sin x)dx$	
	Therefore, $\int_0^\pi(2x^3-3\pi x^2)f(\sin x)\,dx=\int_0^\pi(\pi^3-3\pi^2x)\,f(\sin x)dx$	**B1**
	$\int_0^\pi\frac{1}{(1+\sin x)^2}\,dx=\frac{1}{4}\int_0^\pi\sec^4\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$ $\int_0^\pi\frac{1}{(1+\sin x)^2}\,dx=\frac{1}{4}\int_0^\pi\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)+\tan^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$	**M1**
	$\int_0^\pi\frac{1}{(1+\sin x)^2}\,dx=\frac{1}{4}\left[-2\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)-\frac{2}{3}\tan^3\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]_0^\pi=\frac{4}{3}$	**A1**
	And so, $\int_0^\pi\frac{x}{(1+\sin x)^2}\,dx=\frac{2\pi}{3}$	**B1**
	Therefore, $\int_0^\pi\frac{2x^3-3\pi x^2}{(1+\sin x)^2}\,dx=\pi^3\left(\frac{4}{3}\right)-3\pi^2\left(\frac{2\pi}{3}\right)=-\frac{2}{3}\pi^3.$	**B1**

Question 6

B1	Expression of $\sec^2\theta$ in terms of any other trigonometric functions.
B1	Correct use of a formula such as that for $\cos(A+B)$ to obtain expression with trigonometric functions of $\frac{x}{2}$.
M1	Expanding the squared brackets.
M1	Use of $\sin x \equiv 2\sin\frac{1}{2}x\cos\frac{1}{2}x$ and $\sin^2\frac{1}{2}x + \cos^2\frac{1}{2}x \equiv 1$
A1	Fully justified answer. ***Note that answer is given in the question.***
M1	Any multiple of $\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$.
A1	Correct answer
B1	Deals with change of limits correctly. **AND** Correctly deals with change to integral with respect to u. **Note that both these steps need to be seen – the correct result reached without evidence of these steps should not score this mark.**
B1	Use of $\sin(\pi - x) = \sin x$ (may be just seen within working)
M1	Grouping similar integrals.
A1	Fully justified answer. ***Note that answer is given in the question.***
B1	Evaluation of the integral from (i) with the appropriate limits.
B1	Use of result from (ii) to evaluate required integral.
M1	Attempt to make the substitution.
A1	Substitution all completed correctly.
B1	Rearrange to give something that can represent the required integral on one side.
M1	Use of $\sec^2\theta \equiv 1 + \tan^2\theta$ within integral.
A1	Correct evaluation of this integral.
B1	Correct use of result from part (i).
B1	Correct application of result deduced earlier to reach final answer.

Question 7

(i)	Most likely examples: $x^2 + \left(y \pm \sqrt{r^2 - a^2}\right)^2 = r^2$ and $\left(x \pm \sqrt{r^2 - a^2}\right)^2 + y^2 = r^2$	**M1 M1 A1**
	If $r < a$ then there cannot be two points on the circle that are a distance of $2a$ apart and any two diametrically opposite points on C must be a distance of $2a$ apart.	**B1**
	If $r = a$ then the circle must be the same as C, so there is not exactly 2 points of intersection.	**B1**
(ii)	The distances of the centre of D from the centres of C_1 and C_2 are $\sqrt{r^2 - {a_1}^2}$ and $\sqrt{r^2 - {a_2}^2}$.	**M1 A1 B1**
	If the x-coordinate of the centre of D is x, then the y-coordinate is given by $r^2 - {a_1}^2 = y^2 + (d + x)^2$ and $r^2 - {a_2}^2 = y^2 + (d - x)^2$	**B1 B1**
	Therefore, $(d + x)^2 - (d - x)^2 = (r^2 - {a_1}^2) - (r^2 - {a_2}^2)$	**M1**
	$4dx = {a_2}^2 - {a_1}^2$ and so $x = \frac{{a_2}^2 - {a_1}^2}{4d}$.	**M1 A1**
	Therefore, the y-coordinate of the centre of D satisfies $y^2 = r^2 - {a_1}^2 - \left(d + \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$ and $y^2 = r^2 - {a_2}^2 - \left(d - \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$	**B1**
	So $2y^2 = 2r^2 - {a_1}^2 - {a_2}^2 - \left(d + \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2 - \left(d - \frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$	
	$2y^2 = 2r^2 - {a_1}^2 - {a_2}^2 - 2d^2 - 2\left(\frac{{a_2}^2 - {a_1}^2}{4d}\right)^2$	
	So, $y = \sqrt{r^2 - \frac{{a_1}^2 + {a_2}^2}{2} - d^2 - \left(\frac{{a_2}^2 - {a_1}^2}{4d}\right)^2}$	
	Therefore, $r^2 - \frac{{a_1}^2 + {a_2}^2}{2} - d^2 - \left(\frac{{a_2}^2 - {a_1}^2}{4d}\right)^2 \geq 0$	**B1**
	$16r^2d^2 - 8{a_1}^2d^2 - 8{a_2}^2d^2 - 16d^4 - ({a_2}^2 - {a_1}^2)^2 \geq 0$	**M1 M1**
	$16r^2d^2 \geq 16d^4 + 8{a_1}^2d^2 + 8{a_2}^2d^2 + ({a_2}^2 - {a_1}^2)^2$	
	$16r^2d^2 \geq (4d^2 + {a_1}^2 + {a_2}^2)^2 + ({a_2}^2 - {a_1}^2)^2 - ({a_1}^2 + {a_2}^2)^2$	**M1**
	$16r^2d^2 \geq (4d^2 + {a_1}^2 + {a_2}^2)^2 - 4{a_1}^2{a_2}^2$	**M1**
	$16r^2d^2 \geq (4d^2 + {a_1}^2 + {a_2}^2 - 2a_1a_2)(4d^2 + {a_1}^2 + {a_2}^2 + 2a_1a_2)$	
	$16r^2d^2 \geq (4d^2 + (a_1 - a_2)^2)(4d^2 + (a_1 + a_2)^2)$	**A1**

Question 7

M1	Calculation that the distance between the centres of the circles must be $\sqrt{r^2 - a^2}$.
M1	An example which shows that it is possible for at least one value of r.
A1	Example showing that it is possible for all $r > a$.
B1	Statement that the two intersection points must be a distance $2a$ apart.
B1	Explanation that in the case $r = a$ it would have to be the same circle.
M1	The line joining the centre of C_1 (or C_2) and the radii to a point of intersection form a right angled triangle in each case. (one case)
A1	Use of this to find the distance between centres of circles.
B1	Applying the same to the other circle.
B1	Expression relating the co-ordinates and radii obtained from considering C_1.
B1	Expression relating the co-ordinates and radii obtained from considering C_2.
M1	Elimination of y from the equations.
M1	Either expansion of squared terms or rearrangement to apply difference of two squares.
A1	Expression for x reached. ***Note that answer is given in the question.***
B1	Substitution to find expression for y-coordinate. ***Note that any expression for y in terms of d, r, a_1 and a_2 is sufficient, but it must be expressed as $y = \cdots$, not $y^2 = \cdots$.***
B1	Observation that y^2 must be positive.
	Alternative mark scheme for this may be required once some solutions seen.
M1	Attempt to rearrange the inequality to get $16r^2d^2$ on the left.
M1	Reach a point symmetric in a_1 and a_2.
M1	Reach a combination of squared terms.
M1	Apply difference of two squares to simplify.
A1	Reach the required inequality. ***Note that answer is given in the question.***

Question 8

(i)	Let $\boldsymbol{a}$ be the vector from the centre of C_2 to P.	
	Using similar triangles, the vector from the centre of C_1 to P is $\frac{r_1}{r_2}\boldsymbol{a}$.	**M1 A1**
	Therefore $\frac{r_1}{r_2}\boldsymbol{a} - \boldsymbol{a} = \boldsymbol{x_2} - \boldsymbol{x_1}$, since these are both expressions for the vector from the centre of C_1 to the centre of C_2.	**M1**
	So $\boldsymbol{a} = \frac{r_2}{r_1 - r_2}(\boldsymbol{x_2} - \boldsymbol{x_1})$	**A1**
	The position vector of P is $\boldsymbol{x_2} + \frac{r_2}{r_1 - r_2}(\boldsymbol{x_2} - \boldsymbol{x_1}) = \frac{r_1\boldsymbol{x_2} - r_2\boldsymbol{x_1}}{r_1 - r_2}$	**M1 A1**
(ii)	The position vectors of Q and R will be $\frac{r_3\boldsymbol{x_1} - r_1\boldsymbol{x_3}}{r_3 - r_1}$ and $\frac{r_2\boldsymbol{x_3} - r_3\boldsymbol{x_2}}{r_2 - r_3}$.	**B1**
	Therefore, $\overrightarrow{PQ} = \frac{r_3\boldsymbol{x_1} - r_1\boldsymbol{x_3}}{r_3 - r_1} - \frac{r_1\boldsymbol{x_2} - r_2\boldsymbol{x_1}}{r_1 - r_2} = \frac{\boldsymbol{x_1}[r_3(r_1 - r_2) + r_2(r_3 - r_1)] - \boldsymbol{x_2}[r_1(r_3 - r_1)] - \boldsymbol{x_3}[r_1(r_1 - r_2)]}{(r_3 - r_1)(r_1 - r_2)}$	**M1 A1**
	$\overrightarrow{PQ} = \frac{r_1}{(r_3 - r_1)(r_1 - r_2)}(\boldsymbol{x_1}[r_3 - r_2] + \boldsymbol{x_2}[r_1 - r_3] + \boldsymbol{x_3}[r_2 - r_1])$	**M1 A1**
	Similarly, $\overrightarrow{QR} = \frac{r_3}{(r_2 - r_3)(r_3 - r_1)}(\boldsymbol{x_1}[r_3 - r_2] + \boldsymbol{x_2}[r_1 - r_3] + \boldsymbol{x_3}[r_2 - r_1])$	**M1 A1** **M1 A1**
	Since they are multiples of each other the points P, Q and R must lie on the same straight line.	**B1**
(iii)	Q lies halfway between P and R if $\overrightarrow{PQ} = \overrightarrow{QR}$	**B1**
	Therefore $\frac{r_1}{(r_3 - r_1)(r_1 - r_2)} = \frac{r_3}{(r_2 - r_3)(r_3 - r_1)}$	**M1**
	So, $r_1(r_2 - r_3) = r_3(r_1 - r_2)$	
	Which simplifies to $r_1r_2 + r_2r_3 = 2r_1r_3$	**M1 A1**

Question 8

M1	Identification of similar triangles within the diagram.
A1	Relationship between the two vectors to P.
M1	Equating two expressions for the vector between the centres of the circles.
A1	Correct simplified expression.
M1	Calculation of vector from centre of one circle to P.
A1	Correct position vector for P. ***Note that answer is given in the question.***
B1	Identifying the correct vectors for the foci of the other pairs of circles.
M1	Expression for vector between any two of the foci.
A1	Terms grouped by vector.
M1	Simplification of grouped terms.
A1	Extraction of common factor.
M1 A1 M1 A1	Expression for a vector between a different pair of foci. ***Award marks as same scheme for previous example, but award all four marks for the correct answer written down as it can be obtained by rotating 1, 2 and 3 in the previous answer.***
B1	Statement that they lie on a straight line.
B1	Statement that the two vectors must be equal.
M1	Reduction to statement involving only r terms.
M1	Attempt to simplify expression obtained (if necessary).
A1	Any simplified form.

Question 9

(i)	Taking moments about A:	
	$M_B = 3mga\sin(30+\theta)$	**M1 A1**
	$M_C = 5mga\sin(30-\theta)$	**M1 A1**
	$M_B = M_C$	**B1**
	$5mga(\cos 30\sin\theta + \cos\theta\sin 30) = 3mga(\cos 30\sin\theta - \cos\theta\sin 30)$	**M1**
	$5\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right) = 3\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$	**A1**
	Therefore $4\sqrt{3}\sin\theta = \cos\theta$	**A1**
	Either Use $\sin^2\theta + \cos^2\theta \equiv 1$ and justify choice of positive square root. **Or** Draw right angled triangle such that $\tan\theta = \frac{1}{4\sqrt{3}}$ and calculate the length of the hypotenuse.	**M1**
	$\sin\theta = \frac{1}{7}$	**A1**
(ii)	Let h_1 be the vertical distance of B below A. Let h_2 be the vertical distance of C below A.	
	$h_1 = a\sin\left(\frac{\pi}{3} - \theta\right) = \frac{11}{14}a$	**M1 M1 A1**
	$h_2 = a\sin\left(\frac{\pi}{3} + \theta\right) = \frac{13}{14}a$	**M1 A1**
	If X is the centre of mass of the triangle: $AX = h = \frac{3h_1+5h_2}{8} = \frac{7}{8}a$	**M1 A1**
	Conservation of energy: $4mv^2 \geq 8mg.2h$ for complete revolutions.	**M1 A1**
	Therefore $v_0 = \sqrt{\frac{7ga}{2}}$.	**A1**

Question 9

M1	Attempt to find the moment of B about A.
A1	Correct expression for moment ($\sin(30 + \theta)$ may be replaced by $\cos(60 - \theta)$).
M1	Attempt to find the moment of C about A.
A1	Correct expression for moment ($\sin(30 - \theta)$ may be replaced by $\cos(60 + \theta)$).
B1	Correct statement that these must be equal.
M1	Use of $\sin(A \pm B)$ or $\cos(A \pm B)$ formulae.
A1	Correct values used for $\sin 30$ and $\cos 30$.
A1	Correctly simplified.
M1	Use of a correct method to find the value of $\sin\theta$.
A1	Fully justified solution. If using right angled triangle method then choice of positive root not needed, if choice of positive root not given when applying $\sin^2\theta + \cos^2\theta \equiv 1$ method then M1 A0 should be awarded. ***Note that answer is given in the question.***
M1	Attempt to find h_1.
M1	Correctly deal with sine or cosine term.
A1	Correct value.
M1	Attempt to find h_2.
A1	Correct value.
M1	Combine two values to obtain distance of centre of mass from A.
A1	Correct value
M1	Apply conservation of energy.
A1	Correct inequality.
A1	Correct minimum value.

Question 9 Alternative part (i)

(i)	Let X be the centre of mass of the triangle and let the distance CX be d.	
	Taking moments about X: $5mgd\cos\theta = 3mg(a-d)\cos\theta$	**M1 A1**
	Therefore $5d = 3(a-d)$, so $d = \frac{3}{8}a$.	**A1**
	X must lie on BC and $\angle XAC = 30 - \theta$.	**B1**
	$\sin(30-\theta) = \dfrac{\frac{3}{8}a\cos\theta}{a}$	**M1**
	$\sin 30\cos\theta + \cos 30\sin\theta = \frac{3}{8}\cos\theta$	**M1**
	$\frac{\cos\theta}{8} = \frac{\sqrt{3}\sin\theta}{2}$.	**A1**
	Therefore $\cos\theta = 4\sqrt{3}\sin\theta$ and so $\cos^2\theta = 48\sin^2\theta$	**M1**
	$\sin^2\theta = \frac{1}{49}$, and so (since θ is acute) $\sin\theta = \frac{1}{7}$.	**M1 A1**

M1	Taking moments.
A1	Correct equation.
A1	Correct relationship between d and a.
B1	Identification that X lies on BC and calculation of $\angle XAC$.
M1	Use of sine of identified angle.
M1	Use of $\sin(A-B)$ formula.
A1	Direct relationship between $\sin\theta$ and $\cos\theta$.
M1	Rearrangement and squaring both sides.
M1	Applying $\sin^2\theta + \cos^2\theta \equiv 1$.
A1	Final answer (choice of positive root must be explained). ***Note that answer is given in the question.***

Question 10

	If the length of string from the hole at any moment is l, then $\frac{dl}{dt} = -V$.	**B1**
	The distance, x, from the point beneath the hole satisfies, $h^2 + x^2 = l^2$.	**B1**
	Therefore $\frac{dx}{dt} = \frac{d}{dt}\left((l^2 - h^2)^{\frac{1}{2}}\right) = \frac{1}{2}(l^2 - h^2)^{-\frac{1}{2}} \times 2l\frac{dl}{dt}$.	**M1 A1**
	$\frac{dx}{dt} = -lV(l^2 - h^2)^{-\frac{1}{2}} = -V \times \frac{l}{x}$, and $\frac{l}{x} = \operatorname{cosec}\theta$	**M1**
	Therefore, the speed of the particle is $V\operatorname{cosec}\theta$.	**A1**
	Acceleration: $\frac{d}{dt}(V\operatorname{cosec}\theta) = -V\operatorname{cosec}\theta\cot\theta \times \frac{d\theta}{dt}$	**M1 A1**
	$\sin\theta = \frac{x}{l}$, so $\cos\theta\frac{d\theta}{dt} = \frac{l(-V\operatorname{cosec}\theta) - l\sin\theta(-V)}{l^2} = \frac{V(\sin^2\theta - 1)}{l\sin\theta}$	**M1**
	Therefore $\frac{d\theta}{dt} = -\frac{V}{l}\cot\theta$	**A1**
	The acceleration is $\frac{V^2}{l\sin\theta}\cot^2\theta$	**M1**
	Since $l = h\sec\theta$, the acceleration can be written as $\frac{V^2}{h}\cot^3\theta$.	**M1 A1**
	Horizontally: $T\sin\theta = \frac{mV^2}{h}\cot^3\theta$, so $T = \frac{mV^2}{h}\cot^3\theta\operatorname{cosec}\theta$	**M1 M1 A1**
	The particle will leave the floor when $T\cos\theta = mg$	**M1 A1**
	$\frac{mV^2}{h}\cot^4\theta = mg$ and so $\tan^4\theta = \frac{V^2}{gh}$	**M1 A1**

Question 10

B1	An interpretation of V in terms of other variables (including any newly defined ones).
B1	Any valid relationship between the variables.
M1	Differentiation to find horizontal velocity.
A1	Correct differentiation.
M1	Attempt to eliminate any introduced variables.
A1	Correct result. ***Answers which make clear reference to the speed of the particle in the direction of the string being V.***
M1	Differentiation of speed found in first part.
A1	Correct answer.
M1	Attempt to differentiate to find an expression for $\frac{d\theta}{dt}$.
A1	Correct answer.
M1	Substitution to find expression for acceleration.
M1	Relationship between required variables and any extra variables identified.
A1	Substitution to give answer in terms of correct variables.
M1	Horizontal component of tension.
M1	Application of Newton's second law.
A1	Correct answer.
M1	Vertical component of tension found.
A1	Identification that particle leaves ground when tension is equal to the mass.
M1	Substitution of their value for T.
A1	Rearrangement to give required result. ***Note that answer is given in the question.***

Question 11

(i)	$A(x - a\cos\theta, a\sin\theta)$	**B1 B1**
	Differentiating: $\left(\dot{x} - a(-\sin\theta)\dot{\theta}, a(\cos\theta)\dot{\theta}\right)$	**M1**
	Since B is moving with velocity v and is at the point $(x, 0)$ at time t, $\dot{x} = v$:	
	Velocity of A is $\left(v + a\dot{\theta}\sin\theta, a\dot{\theta}\cos\theta\right)$.	**A1**
(ii)	Initial momentum was mu (horizontally).	**M1**
	Horizontal velocity of C will be the same as that of A, so horizontally the total momentum is given by $mv + 2m\left(v + a\dot{\theta}\sin\theta\right)$	**M1**
	Therefore $3v + 2a\dot{\theta}\sin\theta = u$.	**A1**
	Initial energy was $\frac{1}{2}mu^2$	**M1**
	Total energy is $\frac{1}{2}mv^2 + 2\left(\frac{1}{2}m\left(\left(v + a\dot{\theta}\sin\theta\right)^2 + \left(a\dot{\theta}\cos\theta\right)^2\right)\right)$	**M1 A1**
	Therefore $u^2 = v^2 + 2\left(v^2 + 2av\dot{\theta}\sin\theta + a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2\cos^2\theta\right)$	**M1**
	So $u^2 = 3v^2 + 4av\dot{\theta}\sin\theta + 2a^2\dot{\theta}^2$	
	Substituting $v = \frac{u - 2a\dot{\theta}\sin\theta}{3}$ gives $3u^2 = \left(u - 2a\dot{\theta}\sin\theta\right)^2 + 4a\dot{\theta}\sin\theta\left(u - 2a\dot{\theta}\sin\theta\right) + 6a^2\dot{\theta}^2$	**M1**
	$6a^2\dot{\theta}^2 = 3u^2 - u^2 + 4au\dot{\theta}\sin\theta - 4a^2\dot{\theta}^2\sin^2\theta - 4au\dot{\theta}\sin\theta + 8a^2\dot{\theta}^2\sin^2\theta$	
	$6a^2\dot{\theta}^2 - 4a^2\dot{\theta}^2\sin^2\theta = 2u^2$	
	So, $\dot{\theta}^2 = \frac{u^2}{a^2(3 - 2\sin^2\theta)}$.	**A1**
(iii)	$\dot{\theta}^2 > 0$, so there can only be an instantaneous change of direction in which θ varies at a collision. Since the first collision will be when $\theta = 0$, the second collision must be when $\theta = \pi$.	**B1** **B1**
(iv)	Since horizontal momentum must be mu, $v = 0 \implies 2a\dot{\theta}\sin\theta = u$.	**B1**
	The KE of A must be $\frac{1}{4}mu^2$, so $\frac{1}{2}ma^2\dot{\theta}^2 = \frac{1}{4}mu^2$	**B1**
	$\frac{1}{2}ma^2\dot{\theta}^2 = ma^2\dot{\theta}^2\sin^2\theta$	
	$\sin^2\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$	**M1 A1**
	v is only 0 when θ takes these values and $\dot{\theta}$ is positive as v would need a non-zero value to satisfy $3v + 2a\dot{\theta}\sin\theta = u$ if $\dot{\theta}$ is negative. (The relationship is still true since collisions are elastic).	**B1**

Question 11

B1	Horizontal component.
B1	Vertical component.
M1	Differentiation.
A1	Complete justification, including clear explanation that $\dot{x} = v$. ***Note that answer is given in the question.***
M1	Statement that momentum will be conserved.
M1	Identification that horizontal momentum of A and C will be equal.
A1	Correct equation reached. ***Note that answer is given in the question.***
M1	Statement that energy will be conserved.
M1	Use of symmetry to obtain energy of C (accept answers which simply double the energy of A rather than stating the vertical velocity in opposite direction).
A1	Correct relationship.
M1	Use of $\sin^2\theta + \cos^2\theta \equiv 1$.
M1	Substituting the other relationship to eliminate v.
A1	Correct equation reached. ***Note that answer is given in the question.***
B1	Correct value of θ.
B1	Answer justified.
B1	First equation identified.
B1	Second equation identified.
M1	Solving simultaneously to find θ.
A1	Correct values for θ.
B1	Justified answer that v is not always 0 when θ takes these values.

Question 12

(i)	If a tail occurs then player B must always win before A can achieve the sequence required. Therefore the only way for A to win is if both of the first two tosses are heads.	**B1**
	After the first two tosses are heads it does not matter if more tosses result in heads as the first time tails occurs A will win.	**B1**
	The probability that A wins is therefore $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	**B1**
(ii)	As before, after HH, only A can win.	**B1**
	Similarly, after TT, only C can win.	**B1**
	In all other cases for the first two tosses only B and D will be able to win.	**M1**
	The probabilities for B and D to win must be equal.	**M1**
	The probability of winning is $\frac{1}{4}$ for all of the players.	**A1**
(iii)	If the first two tosses are TT then C must win (as soon as a H occurs), so the probability is 1.	**B1**
	After HT: C must win if the next toss is a T as B needs two Hs to win, but C will win the next time an H occurs.	**M1**
	If the next toss is H, then the position is as if the first two tosses had been TH, and so the probability that C wins from this point is q.	**M1**
	Therefore, $p = \frac{1}{2} \times 1 + \frac{1}{2} \times q$	**A1**
	After HH: If the next toss is H then C will win with probability r.	
	If the next toss is T then C will win with probability p.	**M1**
	Therefore $r = \frac{1}{2}r + \frac{1}{2}p$, and so $p = r$.	**A1**
	After TH: If the next toss is H then player B wins immediately.	
	If the next toss is T then C will win with probability p.	**M1**
	Therefore $q = \frac{1}{2}p$.	**A1**
	Solving the two equations in p and q, gives $p = \frac{2}{3}, q = \frac{1}{3}$	
	From the third equation $r = \frac{2}{3}$	**M1 A1**
	The probability that C wins is $\frac{1}{4}\left(1 + \frac{2}{3} + \frac{1}{3} + \frac{2}{3}\right) = \frac{2}{3}$	**M1 A1**

Question 12

B1	Identifying that A cannot win once a tail has been tossed.
B1	Identifying that A must win once the first two tosses have been heads.
B1	Showing the calculation to reach the answer. ***Note that answer is given in the question.***
B1	Recognising that the situation is unchanged for player A.
B1	Recognising that the same logic applies to player C.
M1	All other cases lead to wins for one of the remaining players.
M1	Recognising that the probabilities must be equal.
A1	Correct statement of the probabilities. **If no marks possible by this scheme award one mark for each probability correctly calculated with supporting working. All four calculated scores 5 marks.**
B1	Explanation that probability must be 1.
M1	Explanation of the case that the next toss is T. **This mark and the next could be awarded for an appropriate tree diagram.**
M1	Explanation of the case that the next toss is H.
A1	Justification of the relationship between p and q. ***Note that answer is given in the question.***
M1	Consideration of one case following HH. **This mark could be awarded for an appropriate tree diagram.**
A1	Establishment of the relationship.
M1	Consideration of one case following TH. **This mark could be awarded for an appropriate tree diagram.**
A1	Establishment of the relationship.
M1	Attempt to solve the simultaneous equations.
A1	Correct values for p, q and r.
M1	Attempt to combine probabilities to obtain overall probability of win.
A1	Correct probability.

Question 13

(i)	$C = \begin{matrix} ky + a(X-y) & \text{for } X > y \\ ky & \text{for } X \leq y \end{matrix}$	B1
	$E(C) = ky + a\int_y^{\infty}(x-y)\lambda e^{-\lambda x}dx$	M1 M1 A1
	Use the substitution $u = x - y$ in the integral:	
	$\int_y^{\infty}(x-y)\lambda e^{-\lambda x}dx = e^{-\lambda y}\int_0^{\infty}u\lambda e^{-\lambda u}du$	B1
	$\int_0^{\infty}u\lambda e^{-\lambda u}du = \left[-ue^{-\lambda u}\right]_0^{\infty} + \int_0^{\infty}e^{-\lambda u}du = \left[-ue^{-\lambda u} - \frac{1}{\lambda}e^{-\lambda u}\right]_0^{\infty} = \frac{1}{\lambda}$	M1
	Therefore $E(C) = ky + \frac{a}{\lambda}e^{-\lambda y}$.	A1
	$\frac{d}{dy}(E(C)) = k - ae^{-\lambda y}$, so the stationary point occurs when $y = \frac{1}{\lambda}\ln\frac{a}{k}$.	M1 A1
	If $\frac{a}{k} > 1$ then choose $y = \frac{1}{\lambda}\ln\frac{a}{k}$ as it is positive.	
	If $\frac{a}{k} \leq 1$ then choose $y = 0$ as the minimum occurs at a negative value of y.	B1
(ii)	$E(C^2) = k^2y^2 + \int_y^{\infty}2aky(x-y)\lambda e^{-\lambda x} + a^2(x-y)^2\lambda e^{-\lambda x}dx$	M1 A1
	Use the substitution $u = x - y$ in the integral:	
	$Integral = e^{-\lambda y}\int_0^{\infty}2akyu\lambda e^{-\lambda u} + a^2u^2\lambda e^{-\lambda u}dx$	B1
	Applying integration done before: $\int_0^{\infty}2akyu\lambda e^{-\lambda u}dx = \frac{2aky}{\lambda}$	
	Using integration by parts: $\int_0^{\infty}a^2u^2\lambda e^{-\lambda u}dx = \left[-a^2u^2e^{-\lambda u}\right]_0^{\infty} + \int_0^{\infty}\frac{2a^2u\lambda e^{-\lambda u}}{\lambda}dx$	M1 A1
	and, applying the integration already completed, $\int_0^{\infty}\frac{2a^2ue^{-\lambda u}}{\lambda}dx = \frac{2a^2}{\lambda^2}$.	
	Therefore $E(C^2) = k^2y^2 + \frac{2aky}{\lambda}e^{-\lambda y} + \frac{2a^2}{\lambda^2}e^{-\lambda y}$.	A1
	$\text{Var}(C^2) = E(C^2) - E(C)^2$	M1
	$\text{Var}(C^2) = k^2y^2 + \frac{2aky}{\lambda}e^{-\lambda y} + \frac{2a^2}{\lambda^2}e^{-\lambda y} - \left(ky + \frac{a}{\lambda}e^{-\lambda y}\right)^2$.	
	$\text{Var}(C^2) = \frac{a^2}{\lambda^2}\left(2e^{-\lambda y} - e^{-2\lambda y}\right)$.	A1
	$\frac{d}{dy}\left(\text{Var}(C^2)\right) = \frac{2a^2}{\lambda}e^{-\lambda y}\left(e^{-\lambda y} - 1\right)$	M1
	For $y > 0$, $\frac{d}{dy}\left(\text{Var}(C^2)\right) < 0$, so the variance decreases as y increases.	A1

Question 13

B1	Statement of random variable.
M1	Any correct term in expectation (allow ky multiplied by an attempt at the probability for not needing any extra costs).
M1	Correct integral stated (allow $-y$ missing).
A1	Fully correct statement.
	May be altered to accommodate other methods once solutions seen.
B1	Substitution performed correctly.
M1	Integration by parts used to calculate integral.
A1	Correctly justified solution. ***Note that answer is given in the question.***
M1	Differentiation to find minimum point.
A1	Correct identification of point.
B1	Both cases identified with the solutions stated.
M1	Attempt at $E(C^2)$ (at least two terms correct).
A1	Correct statement of $E(C^2)$.
B1	Substitution performed correctly.
M1	Applying integration by parts.
A1	Correct integration.
A1	Correct expression for $E(C^2)$.
M1	Use of $\mathrm{Var}(C^2) = E(C^2) - E(C)^2$
A1	Correct simplified form for $\mathrm{Var}(C^2)$
M1	Differentiation of $\mathrm{Var}(C^2)$.
A1	Correct interpretation. ***Note that answer is given in the question.***

1. (i)

$$I_n - I_{n+1} = \int_0^\infty \frac{1}{(1+u^2)^n}du - \int_0^\infty \frac{1}{(1+u^2)^{n+1}}du = \int_0^\infty \frac{1+u^2-1}{(1+u^2)^{n+1}}du$$

$$= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}}du$$

B1

$$= \int_0^\infty u\frac{u}{(1+u^2)^{n+1}}du = \left[u\frac{-1}{2n(1+u^2)^n}\right]_0^\infty - \int_0^\infty \frac{-1}{2n(1+u^2)^n}du$$

integrating by parts **M1 A1**

$$= 0 + \frac{1}{2n}\int_0^\infty \frac{1}{(1+u^2)^n}du = \frac{1}{2n}I_n$$

A1* **(4)**

$I_{n+1} = I_n - \frac{1}{2n}I_n = \frac{2n-1}{2n}I_n$ **M1**

$= \frac{(2n-1)(2n-3)\ldots(1)}{(2n)(2n-2)\ldots(2)}I_1$ **M1**

$I_1 = \int_0^\infty \frac{1}{(1+u^2)}du = [\tan^{-1}u]_0^\infty = \frac{\pi}{2}$ **B1**

$\frac{(2n-1)(2n-3)\ldots(1)}{(2n)(2n-2)\ldots(2)} = \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots(2)(1)}{[(2n)(2n-2)\ldots(2)]^2} = \frac{(2n)!}{[2^n n!]^2} = \frac{(2n)!}{2^{2n}(n!)^2}$ **M1**

Thus $I_{n+1} = \frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2} = \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$ **A1*** **(5)**

(ii)

$$J = \int_0^\infty f((x-x^{-1})^2)dx = \int_\infty^0 f((u^{-1}-u)^2).-u^{-2}du = \int_0^\infty x^{-2}f((x-x^{-1})^2)dx$$

using the substitution $u = x^{-1}$, $\frac{du}{dx} = -x^{-2}$ and then the substitution $u = x$, $\frac{du}{dx} = 1$ **M1A1***

$$2J = \int_0^\infty f((x-x^{-1})^2)dx + \int_0^\infty x^{-2}f((x-x^{-1})^2)dx = \int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx$$

So $J = \frac{1}{2}\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx$ **M1A1***

Using the substitution $u = x - x^{-1}$, $\frac{du}{dx} = 1 + x^{-2}$,

$$J=\frac{1}{2}\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx=\frac{1}{2}\int_{-\infty}^{\infty} f(u^2)\,du=\int_0^\infty f(u^2)du$$ M1A1* (6)

(iii)

$$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}dx=\int_0^\infty \frac{x^{-2}}{(x^2-1+x^{-2})^n}dx=\int_0^\infty \frac{x^{-2}}{((x-x^{-1})^2+1)^n}dx$$ **M1A1**

$$=\int_0^\infty \frac{1}{(u^2+1)^n}du$$ **M1**

So

$$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}dx=\int_0^\infty \frac{1}{(u^2+1)^n}du=I_n$$ **M1**

$$=\frac{\big(2(n-1)\big)!\pi}{2^{2(n-1)+1}\big((n-1)!\big)^2}=\frac{(2n-2)!\pi}{2^{2n-1}\big((n-1)!\big)^2}$$ **A1 (5)**

2. (i) True. **B1**

$m = 1000$ **B1**

If $n \geq 1000$, then $1000 \leq n$, so $1000n \leq n^2$, i.e. $(1000n) \leq (n^2)$ **M1A1 (4)**

(ii) False. **B1**

E. G. Let $s_n = 1$ and $t_n = 2$ for n odd, and $s_n = 2$ and $t_n = 1$ for n even. **B1**

Then $\nexists m$ for which for $n \geq m$, $s_n \leq t_n$, nor $t_n \leq s_n$ **M1**

So it is not the case that $(s_n) \leq (t_n)$, but nor is it the case that $(t_n) \leq (s_n)$ **A1 (4)**

(iii) True. **B1**

$(s_n) \leq (t_n)$ means that there exists a positive integer, say m_1, for which for $n \geq m_1$, $s_n \leq t_n$.
E1

$(t_n) \leq (u_n)$ means that there exists a positive integer, say m_2, for which for $n \geq m_2$, $t_n \leq u_n$.
E1

Then if $m = \max(m_1, m_2)$, **B1**

for $n \geq m$, $s_n \leq t_n \leq u_n$, and so $(s_n) \leq (u_n)$ **A1 (5)**

(iv) True. **B1**

$m = 4$ **B1**

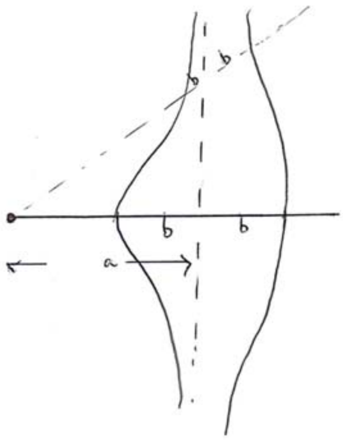
Assume $k^2 \leq 2^k$ for some value $k \geq 4$. **B1**

Then $(k+1)^2 = \left(\frac{k+1}{k}\right)^2 k^2 = \left(1 + \frac{1}{k}\right)^2 k^2 \leq \left(1 + \frac{1}{4}\right)^2 k^2 = \frac{25}{16} k^2 < 2k^2 \leq 2 \times 2^k = 2^{k+1}$
M1A1

$4^2 = 2^4$ **B1**

so by the principle of mathematical induction, $n^2 \leq 2^n$ for $n \geq 4$, and thus $(n^2) \leq (2^n)$ **A1 (7)**

3. (i)



Symmetry about initial line **G1**

Two branches **G1**

Shape and labelling **G1 (3)**

If $|r - a\sec\theta| = b$, then $r - a\sec\theta = b$ or $r - a\sec\theta = -b$

So $r = a\sec\theta + b$ or $r = a\sec\theta - b$ **M1A1**

If $\sec\theta < 0$, $a\sec\theta + b < -a + b < 0$ as $a > b$ and $a\sec\theta - b < -a - b < 0$ as a and b are both positive, and thus in both cases, $r < 0$ which is not permitted. **B1**

If $\sec\theta > 0$, $a\sec\theta + b > a + b > 0$ and $a\sec\theta - b > a - b > 0$ giving $r > 0$

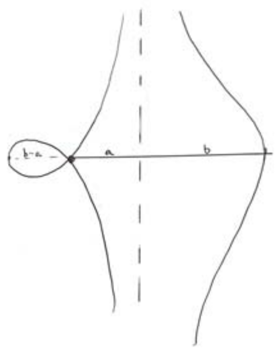
so $\sec\theta > 0$ as required. **B1** **(4)**

So $r = a\sec\theta \pm b$, thus points satisfying (*) lie on a certain conchoid of Nicomedes with *A* being the pole (origin), **B1**

d being *b*, **B1**

and L being the line $r = a\sec\theta$. **B1 (3)**

(ii)



Symmetry about initial line **G1**

Two branches **G1**

Loop, shape and labelling **G1**

If $a < b$, then the curve has two branches, $r = a\sec\theta + b$ with $\sec\theta > 0$ and $r = a\sec\theta + b$ with $\sec\theta < 0$, the endpoints of the loop corresponding to $\sec\theta = \frac{-b}{a}$. **B1 (4)**

In the case $a = 1$ and $b = 2$, $\sec\theta = \frac{-2}{1} = -2$ so $\theta = \pm\frac{2\pi}{3}$

Area of loop

$= 2 \times \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}(\sec\theta + 2)^2 d\theta$ **M1A1**

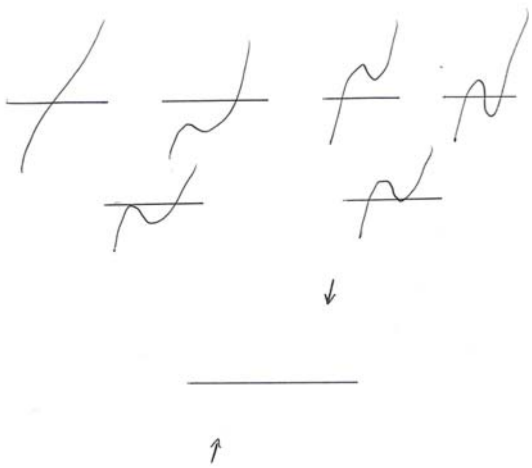
$= \int_{\frac{2\pi}{3}}^{\pi}\sec^2\theta + 4\sec\theta + 4\, d\theta = [\tan\theta + 4\ln|\sec\theta + \tan\theta| + 4\theta]_{\frac{2\pi}{3}}^{\pi}$ **M1A1**

$= 4\pi - \left(-\sqrt{3} + 4\ln\left|-2 - \sqrt{3}\right| + \frac{8\pi}{3}\right) = \frac{4\pi}{3} + \sqrt{3} - 4\ln\left|2 + \sqrt{3}\right|$ **M1A1 (6)**

4. (i) $y = z^3 + az^2 + bz + c$ is continuous.

For $z \to -\infty, y \to -\infty$ and for $z \to \infty, y \to \infty$. **B1**

So the sketch of this graph must be one of the following:-



B1

Hence, it must intersect the z axis at least once, and so there is at least one real root of

$z^3 + az^2 + bz + c = 0$ **B1 (3)**

(ii) $z^3 + az^2 + bz + c = (z - z_1)(z - z_2)(z - z_3)$ **M1**

Thus $a = (-z_1 - z_2 - z_3) = -S_1$ **A1**

$b = (z_2 z_3 + z_3 z_1 + z_1 z_2) = \frac{(z_1+z_2+z_3)^2 - ({z_1}^2+{z_2}^2+{z_3}^2)}{2} = \frac{{S_1}^2 - S_2}{2}$ **A1**

and, as ${z_1}^3 + a{z_1}^2 + bz_1 + c = 0$, ${z_2}^3 + a{z_2}^2 + bz_2 + c = 0$, ${z_3}^3 + a{z_3}^2 + bz_3 + c = 0$

adding these three equations we have,

$({z_1}^3 + {z_2}^3 + {z_3}^3) + a({z_1}^2 + {z_2}^2 + {z_3}^2) + b(z_1 + z_2 + z_3) + 3c = 0$ **M1**

(Alternatively,

$(z_1 + z_2 + z_3)^3 =$

$({z_1}^3 + {z_2}^3 + {z_3}^3) + 3({z_1}^2 z_2 + {z_2}^2 z_3 + {z_3}^2 z_1 + {z_1}^2 z_3 + {z_2}^2 z_1 + {z_3}^2 z_2) + 6z_1 z_2 z_3$

$({z_1}^2 + {z_2}^2 + {z_3}^2)(z_1 + z_2 + z_3) = ({z_1}^3 + {z_2}^3 + {z_3}^3) + ({z_1}^2 z_2 + {z_2}^2 z_3 + {z_3}^2 z_1 + {z_1}^2 z_3 + {z_2}^2 z_1 +$
${z_3}^2 z_2)$)

So $S_3 - S_1 S_2 + \frac{{S_1}^2 - S_2}{2} S_1 + 3c = 0$ **M1**

Thus $6c = \left(3S_1 S_2 - {S_1}^3 - 2S_3\right)$ **A1* (6)**

(iii) Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ for $k = 1, 2, 3$ **M1**

Then ${z_k}^2 = {r_k}^2(\cos 2\theta_k + i\sin 2\theta_k)$ and ${z_k}^3 = {r_k}^3(\cos 3\theta_k + i\sin 3\theta_k)$ by de Moivre **M1**

As

$$\sum_{k=1}^{3} r_k \sin\theta_k = 0$$

$$\sum_{k=1}^{3} {r_k}^2 \sin 2\theta_k = 0$$

$$\sum_{k=1}^{3} {r_k}^3 \sin 3\theta_k = 0$$

$$Im\left(\sum_{k=1}^{3} z_k\right) = 0$$

$$Im\left(\sum_{k=1}^{3} {z_k}^2\right) = 0$$

$$Im\left(\sum_{k=1}^{3} {z_k}^3\right) = 0$$

and so S_1, S_2, and S_3 are real, **M1**

and therefore so are a, b, and c **A1**

Hence, as z_1, z_2, and z_3 are the roots of $z^3 + az^2 + bz + c = 0$ with a, b, and c real, by part (i), at least one of z_1, z_2, and z_3 is real. **M1**

So for at least one value of k, $r_k(\cos\theta_k + i\sin\theta_k)$ is real and thus, $\sin\theta_k = 0$,

and as $-\pi < \theta_k < \pi$, $\theta_k = 0$ as required. **A1 (6)**

If $\theta_1 = 0$ then z_1 is real. z_2 and z_3 are the roots of $(z - z_2)(z - z_3) = 0$

which is $z^2 + (-z_2 - z_3)z + z_2 z_3 = 0$ (say $z^2 + pz + q = 0$)

$p = -z_2 - z_3 = a + z_1$ and $q = z_2 z_3 = -\frac{c}{z_1}$ and so the quadratic of which z_2 and z_3 are the roots has real coefficients. Thus $z_2,\ z_3 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$. ($z_1 \neq 0$ because $r_k > 0$) **B1**

If $p^2 - 4q < 0$, **M1**

Thus $\cos\theta_2 = \cos\theta_3$, and so $\theta_2 = \pm\theta_3$, as $-\pi < \theta_k < \pi$.

But $\sin\theta_2 = -\sin\theta_3$ and so $\theta_2 = -\theta_3$. **M1 A1**

If $p^2 - 4q \geq 0$, then z_2 and z_3 are real roots, so $\sin\theta_2 = \sin\theta_3 = 0$, and thus $\theta_2 = \theta_3 = 0$, so $\theta_2 = -\theta_3$. **B1 (5)**

5. (i) Having assumed that $\sqrt{2}$ is rational (step 1), $\sqrt{2} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$ **B1**

Thus from the definition of S (step 2), as $q \in \mathbb{Z}$ and $\sqrt{2} = q \times {}^{p}/_{q} = p \in \mathbb{Z}$, so $q \in S$ proving step 3. **B1 (2)**

If $k \in S$, then k is an integer and $k\sqrt{2}$ is an integer. **B1**

So $(\sqrt{2} - 1)k = k\sqrt{2} - k$ is an integer, **B1**

and $(\sqrt{2} - 1)k\sqrt{2} = 2k - k\sqrt{2}$ which is an integer and so $(\sqrt{2} - 1)k \in S$ proving step 5. **B1 (3)**

$1 < \sqrt{2} < 2$ and so **M1**

$0 < \sqrt{2} - 1 < 1$, and thus $0 < (\sqrt{2} - 1)k < k$ **A1**

and thus this contradicts step 4 that k is the smallest positive integer in S as $(\sqrt{2} - 1)k$ has been shown to be a smaller positive integer and is in S . **A1 (3)**

(ii) If $2^{2/3}$ is rational, then $2^{2/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$

So $\left(2^{2/3}\right)^2 = \left({}^{p}/_{q}\right)^2$, that is $2^{4/3} = {}^{p^2}/_{q^2}$, which can be written $2 \times 2^{1/3} = {}^{p^2}/_{q^2}$ **M1**

and hence $2^{1/3} = {}^{p^2}/_{2q^2}$ proving that $2^{1/3}$ is rational. **A1**

If $2^{1/3}$ is rational, then $2^{1/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$ **M1**

and so $2^{2/3} = {}^{p^2}/_{q^2}$ proving that $2^{2/3}$ is rational and that $2^{1/3}$ is rational only if $2^{2/3}$ is rational. **A1 (4)**

Assume that $2^{1/3}$ is rational.

Define the set T to be the set of positive integers with the following property: n is in T if and only if $n2^{1/3}$ and $n2^{2/3}$ are integers. **B1**

The set T contains at least one positive integer as if $2^{1/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$, then $q^2 2^{1/3} = q^2 \times {}^{p}/_{q} = pq \in \mathbb{Z}$ and $q^2 2^{2/3} = q^2 \times {}^{p^2}/_{q^2} = p^2 \in \mathbb{Z}$, so $q^2 \in T$. **M1A1**

Define t to be the smallest positive integer in T. Then $t2^{1/3}$ and $t2^{2/3}$ are integers. **B1**

Consider $t\left(2^{2/3} - 1\right)$. $t\left(2^{2/3} - 1\right) = t2^{2/3} - t$ which is the difference of two integers and so is itself an integer. $t\left(2^{2/3} - 1\right) \times 2^{1/3} = 2t - t2^{1/3}$ which is an integer,

and $t\left(2^{2/3} - 1\right) \times 2^{2/3} = 2^{4/3}t - t2^{2/3} = 2 \times 2^{1/3}t - t2^{2/3}$ which is an integer.

Thus $t\left(2^{2/3}-1\right)$ is in T. **M1A1**

$1 < 2^{2/3} < 2$ and so $0 < 2^{2/3} - 1 < 1$, and thus $0 < t\left(2^{2/3}-1\right) < t$, and thus this contradicts that t is the smallest positive integer in T as $t\left(2^{2/3}-1\right)$ has been shown to be a smaller positive integer and is in T. **M1A1 (8)**

6. (i) $w, z \in \mathbb{R} \Longrightarrow u, v \in \mathbb{R}$ **B1**

For $w, z \in \mathbb{R}$, we require to solve $w + z = u$, $w^2 + z^2 = v$ **M1**

$$w^2 + (u - w)^2 = v$$

$$2w^2 - 2uw + (u^2 - v) = 0$$

$$w = \frac{2u \pm \sqrt{4u^2 - 8u^2 + 8v}}{4} = \frac{u \pm \sqrt{2v - u^2}}{2}$$

$$z = \frac{u \mp \sqrt{2v - u^2}}{2}$$

M1A1

So for , $z \in \mathbb{R}$, as $u = w + z$ must be real, $v = w^2 + z^2$ must be real, and $2v - u^2 \geq 0$

i.e. $u^2 \leq 2v$ **B1* (5)**

(ii) $u = w + z \Longrightarrow u^2 = w^2 + z^2 + 2wz$ so if $w^2 + z^2 - u^2 = -\frac{2}{3}$, then $-2wz = -\frac{2}{3}$

so $3wz = 1$ **M1A1**

$$w^3 + z^3 = (w + z)(w^2 + z^2 - wz) = u(u^2 - 3wz) = u(u^2 - 1)$$

M1A1

Thus if $w^3 + z^3 - \lambda u = -\lambda$, $u(u^2 - 1) = \lambda(u - 1)$ **M1A1**

Thus $(u - 1)(u(u + 1) - \lambda) = 0$, **M1**

$(u - 1)(u^2 + u - \lambda) = 0$ **M1A1**

Thus $u = 1$ or $u = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}$

So as $\lambda \in \mathbb{R}$ and $\lambda > 0$, the values of u are real. **B1**

There are three distinct values of u unless $\frac{-1 \pm \sqrt{1 + 4\lambda}}{2} = 1$ in which case $\pm\sqrt{1 + 4\lambda} = 3$, i.e. $\lambda = 2$

M1A1 (12)

For $w, z \in \mathbb{R}$, from (i) we require $u \in \mathbb{R}$ which it is, $u^2 - \frac{2}{3} \in \mathbb{R}$ which it is, and $u^2 \leq 2\left(u^2 - \frac{2}{3}\right)$ in other words $u^2 \geq \frac{4}{3}$. **M1**

So w and z need not be real. A counterexample would be $u = 1$ **B1**

for then $w + z = 1$, $w^2 + z^2 = \frac{1}{3}$, so $w^2 + (1 - w)^2 = \frac{1}{3}$, i.e. $2w^2 - 2w + \frac{2}{3} = 0$ in which case the discriminant is $-\frac{4}{3} < 0$ so $w \notin \mathbb{R}$. **B1 (3)**

7. $D^2x^a = D(D(x^a)) = D\left(x\frac{d}{dx}(x^a)\right) = D(xax^{a-1})$ **M1**

$= D(ax^a) = x\frac{d}{dx}(ax^a) = xa^2x^{a-1} = a^2x^a$ **M1A1 (3)**

(i) Suppose $D^kP(x)$ is a polynomial of degree r i.e. $D^kP(x) = a_rx^r + a_{r-1}x^{r-1} + \cdots + a_0$ for some integer k . **B1**

Then $D^{k+1}P(x) = D(a_rx^r + a_{r-1}x^{r-1} + \cdots + a_0) = x\frac{d}{dx}(a_rx^r + a_{r-1}x^{r-1} + \cdots + a_0)$

$= x(ra_rx^{r-1} + (r-1)a_{r-1}x^{r-2} + \cdots + a_1) = ra_rx^r + (r-1)a_{r-1}x^{r-1} + \cdots + a_1x$

which is a polynomial of degree r . **M1A1**

Suppose $P(x) = b_rx^r + b_{r-1}x^{r-1} + \cdots + b_0$, then

$DP(x) = x\frac{d}{dx}(b_rx^r + b_{r-1}x^{r-1} + \cdots + b_0) = rb_rx^r + (r-1)b_{r-1}x^{r-1} + \cdots + b_1x$ so the result is true for $n = 1$, **M1A1**

and we have shown that if it is true for $n = k$, it is true for $n = k + 1$. Hence by induction, it is true for any positive integer . **B1 (6)**

(ii) Suppose $D^k(1-x)^m$ is divisible by $(1-x)^{m-k}$ i.e. $D^k(1-x)^m = f(x)(1-x)^{m-k}$ for some integer k , with $k < m - 1$. **B1**

Then $D^{k+1}(1-x)^m = D(f(x)(1-x)^{m-k}) = x\frac{d}{dx}(f(x)(1-x)^{m-k})$

$= x(f'^{(x)}(1-x)^{m-k} - (m-k)f(x)(1-x)^{m-k-1})$

$= x(1-x)^{m-k-1}\left(f'^{(x)}(1-x) - (m-k)f(x)\right)$ which is divisible by $(1-x)^{m-(k+1)}$. **M1A1**

$D(1-x)^m = x\frac{d}{dx}((1-x)^m) = -mx(1-x)^{m-1}$ so result is true for $n = 1$. **M1A1**

We have shown that if it is true for $n = k$, it is true for $n = k + 1$. Hence by induction, it is true for any positive integer $< m$. **B1 (6)**

(iii)

$(1-x)^m = \sum_{r=0}^{m}\binom{m}{r}(-x)^r = \sum_{r=0}^{m}(-1)^r\binom{m}{r}x^r$ **M1**

So

$D^n(1-x)^m = \sum_{r=0}^{m}(-1)^r\binom{m}{r}D^nx^r = \sum_{r=0}^{m}(-1)^r\binom{m}{r}r^nx^r$ **M1A1**

But by (ii), $D^n(1-x)^m$ is divisible by $(1-x)^{m-n}$ and so $D^n(1-x)^m = g(x)(1-x)^{m-n}$, and thus if $x = 1$, $D^n(1-x)^m = 0$, and hence

$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0$ **M1A1* (5)**

8. (i) $x = r\cos\theta \implies \frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$ **M1A1**

and $y = r\sin\theta \implies \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$ **M1A1**

Thus $(y+x)\frac{dy}{dx} = y - x$ becomes $(r\sin\theta + r\cos\theta)\frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = r\sin\theta - r\cos\theta$ **M1**

That is $(\sin\theta + \cos\theta)\left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right) = (\sin\theta - \cos\theta)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$

as $r > 0,\ r \neq 0$

Multiplying out and collecting like terms gives

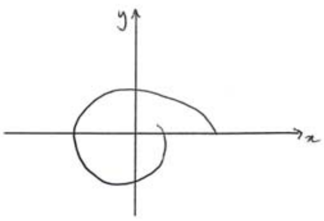
$$r(\cos^2\theta + \sin^2\theta) + \frac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) = 0$$

which is $r + \frac{dr}{d\theta} = 0$. **M1A1* (7)**

So $re^{\theta} + \frac{dr}{d\theta}e^{\theta} = 0$ **M1**

and thus $re^{\theta} = k$, **A1**

$r = ke^{-\theta}$ **A1**



G1 (4)

(or alternatively $\int \frac{1}{r}dr = \int -d\theta$ **M1** so $\ln|r| = -\theta + c$ **A1** and hence $r = ke^{-\theta}$ **A1**)

(ii) $\left(y + x - x(x^2+y^2)\right)\frac{dy}{dx} = y - x - y(x^2+y^2)$

becomes $(r\sin\theta + r\cos\theta - r^3\cos\theta)\frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = r\sin\theta - r\cos\theta - r^3\sin\theta$

that is

$(\sin\theta + \cos\theta - r^2\cos\theta)\left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right) = (\sin\theta - \cos\theta - r^2\sin\theta)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$

Multiplying out and collecting like terms gives

$r\left(\cos^2\theta + \sin^2\theta - r^2(\cos^2\theta + \sin^2\theta)\right) + \frac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) = 0$ **M1**

which is $r - r^3 + \frac{dr}{d\theta} = 0$. **A1**

$\int \frac{1}{r^3 - r} dr = \int d\theta$

$\int \frac{1}{r^3 - r} dr = \int \frac{1}{r(r^2-1)} dr = \int \frac{1}{r(r-1)(r+1)} dr = \int d\theta$ **M1**

So $\int d\theta = \int \frac{1/2}{r-1} + \frac{-1}{r} + \frac{1/2}{r+1} dr$ **A1**

$\theta + k = \frac{1}{2}\ln\left|\frac{(r-1)(r+1)}{r^2}\right|$ **A1**

So

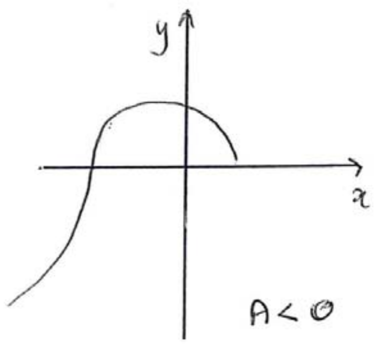
$$\left|\frac{r^2 - 1}{r^2}\right| = Ce^{2\theta}$$

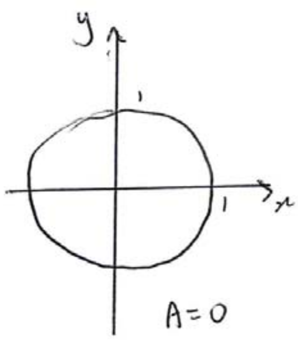
with $C > 0$

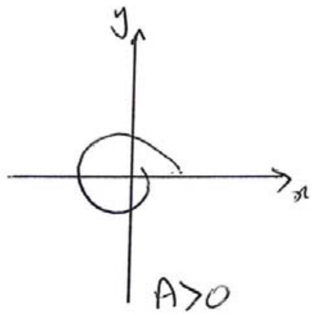
$$r^2 = \frac{1}{1 \mp Ce^{2\theta}}$$

that is

$r^2 = \frac{1}{1 + Ae^{2\theta}}$ **A1***







G1 G1 G1 (9)

9. If the initial position of P is , then at time t , $OP^2 = a^2 + x^2$, so conserving energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + \frac{\lambda}{2a}\left(\sqrt{a^2+x^2} - a\right)^2$$

M1 A1 A1

Thus,

$$\dot{x}^2 = v^2 - \frac{\lambda}{ma}\left(\sqrt{a^2+x^2} - a\right)^2$$

M1

i.e.

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{a^2+x^2} - a\right)^2$$

A1* (5)

The greatest value, x_0 , attained by x , occurs when $\dot{x} = 0$. **M1**

Thus $v^2 = k^2\left(\sqrt{a^2+{x_0}^2} - a\right)^2$

So $\sqrt{a^2+{x_0}^2} - a = \frac{v}{k}$ (negative root discounted as all quantities are positive)

Thus

$${x_0}^2 = \left(\frac{v}{k} + a\right)^2 - a^2 = \frac{v^2}{k^2} + \frac{2av}{k}$$

and

$$x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}}$$

M1 A1 (3)

As

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{a^2+x^2} - a\right)^2$$

differentiating with respect to t

$$2\dot{x}\ddot{x} = -2k^2\left(\sqrt{a^2+x^2} - a\right)\frac{1}{2}(a^2+x^2)^{\frac{-1}{2}}2x\dot{x}$$

M1 A1

Thus

$$\ddot{x} = -xk^2\frac{\left(\sqrt{a^2+x^2} - a\right)}{\sqrt{a^2+x^2}}$$

A1

So when $x = x_0$, the acceleration of P is

$$-x_0 k^2 \frac{\frac{v}{k}}{\frac{v}{k}+a} = -\sqrt{\frac{v^2}{k^2}+\frac{2av}{k}}\,k^2 \frac{\frac{v}{k}}{\frac{v}{k}+a} = -\frac{kv\sqrt{v^2+2akv}}{v+ak}$$

M1 A1 (5)

$$\dot{x} = \left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}$$

That is

$$\frac{dx}{dt} = \left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}$$

and thus

$$\int_0^{\tau/4} dt = \int_0^{x_0} \frac{1}{\left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}}\,dx$$

where τ is the period. **M1 A1**

So

$$\tau = 4\int_0^{x_0} \frac{1}{\left[v^2 - k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}}\,dx$$

$$\tau = \frac{4}{v}\int_0^{\sqrt{\frac{v^2}{k^2}+\frac{2av}{k}}} \frac{1}{\left[1-\frac{k^2\left(\sqrt{a^2+x^2}-a\right)^2}{v^2}\right]^{\frac{1}{2}}}\,dx$$

Let

$$u^2 = \frac{k\left(\sqrt{a^2+x^2}-a\right)}{v}$$

B1

then

$$a^2 + x^2 = \left(\frac{vu^2}{k}+a\right)^2$$

and so

$$x^2 = \frac{v^2\,u^4 + 2kavu^2}{k^2}$$

$$x = \sqrt{2kav}\frac{u}{k}\left(1 + \frac{v}{2ka}u^2\right)^{\frac{1}{2}} \approx \sqrt{2kav}\frac{u}{k}$$

as $v \ll ka$

Thus

$$\frac{dx}{du} \approx \frac{1}{k}\sqrt{2kav}$$

M1A1

and so

$$\tau \approx \frac{4}{v}\int_0^1 \frac{1}{\sqrt{1-u^4}}\frac{1}{k}\sqrt{2kav}du = \sqrt{\frac{32a}{kv}}\int_0^1 \frac{1}{\sqrt{1-u^4}}du$$

as required. **M1 A1* (7)**

10. The position vector of the upper particle is

$$\begin{pmatrix} x + a\sin\theta \\ y + a\cos\theta \end{pmatrix}$$

B1 B1

so differentiating with respect to time, its velocity is

$$\begin{pmatrix} \dot{x} + a\,\dot{\theta}\cos\theta \\ \dot{y} - a\,\dot{\theta}\sin\theta \end{pmatrix}$$

E1* (3)

Its acceleration, by differentiating with respect to time, is thus

$$\begin{pmatrix} \ddot{x} + a\,\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\,\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix}$$

M1 A1 A1

so by Newton's second law resolving horizontally and vertically

$$\begin{pmatrix} -T\sin\theta \\ -T\cos\theta - mg \end{pmatrix} = m\begin{pmatrix} \ddot{x} + a\,\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\,\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix}$$

M1 A1

That is

$$m\begin{pmatrix} \ddot{x} + a\,\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\,\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The other particle's equation is

$$m\begin{pmatrix} \ddot{x} - a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ \ddot{y} + a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

B1 (6)

Adding these two equations we find

$$2m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -2mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

i.e. $\ddot{x} = 0$ and $\ddot{y} = -g$ **M1 A1***

Thus

$$m\begin{pmatrix} -a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

i.e. $m\left(-a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta\right) = T\sin\theta$ and $m\left(a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta\right) = T\cos\theta$

Multiplying the second of these by $\sin\theta$ and the first by $\cos\theta$ and subtracting,

$ma\ddot{\theta} = 0$ and so $\ddot{\theta} = 0$. **M1A1* (4)**

Thus $\dot{\theta} = a\ constant$ and as initially $2a\dot{\theta} = u$, $\dot{\theta} = \frac{u}{2a}$ **M1 A1**

Therefore the time to rotate by $\frac{1}{2}\pi$ is given by $\tau\dot{\theta} = \frac{1}{2}\pi$, so $\tau = \frac{1}{2}\pi \div \frac{u}{2a} = \frac{\pi a}{u}$ **A1**

As $\ddot{y} = -g$ and initially $\dot{y} = v$, at time , $\dot{y} = v - gt$, and so $y = vt - \frac{1}{2}gt^2 + h$ as the centre of the rod is initially h above the table. **M1 A1**

Hence, given the condition that the particles hit the table simultaneously,

$$0 = v\,\pi a/u - 1/2\ g(\pi a/u)^2 + h$$

.

Hence $0 = 2\pi uva - \pi^2a^2g + 2hu^2$, or $2hu^2 = \pi^2a^2g - 2\pi uva$ as required. **M1 A1* (7)**

11. (i) Suppose that the force exerted by P on the rod has components X perpendicular to the rod and Y parallel to the rod. Then taking moments for the rod about the hinge, $Xd = 0$, **M1**

which as $d \neq 0$ yields $X = 0$ and hence the force exerted on the rod by P is parallel to the rod. **A1* (2)**

Resolving perpendicular to the rod for P, $mg \sin\alpha = m(r - d\sin\alpha)\omega^2 \cos\alpha$ **M1 A1**

Dividing by $m\omega^2 \sin\alpha$, $\frac{g}{\omega^2} = (r - d\sin\alpha)\cot\alpha$

That is $a = r\cot\alpha - d\cos\alpha$ or in other words $r\cot\alpha = a + d\cos\alpha$ as required. **M1 A1* (4)**

The force exerted by the hinge on the rod is along the rod towards P, **B1**

and if that force is F, then resolving vertically for P, $F\cos\alpha = mg$ **M1 A1**

so $F = mg\sec\alpha$. **A1 (4)**

(ii) Suppose that the force exerted by m_1 on the rod has component X_1 perpendicular to the rod towards the axis, that the force exerted by m_2 on the rod has component X_2 perpendicular to the rod towards the axis, **B1**

then resolving perpendicular to the rod for m_1, $m_1 g\sin\beta + X_1 = m_1(r - d_1\sin\beta)\omega^2\cos\beta$ **M1A1**

and similarly for m_2, $m_2 g\sin\beta + X_2 = m_2(r - d_2\sin\beta)\omega^2\cos\beta$

M1A1

Taking moments for the rod about the hinge, $X_1 d_1 + X_2 d_2 = 0$ **M1A1**

So multiplying the first equation by d_1, the second by d_2 and adding we have

$$m_1 g\, d_1 \sin\beta + m_2 d_2 g\sin\beta = m_1 d_1(r - d_1\sin\beta)\omega^2\cos\beta + m_2 d_2(r - d_2\sin\beta)\omega^2\cos\beta$$

Dividing by $(m_1 d_1 + m_2 d_2)\omega^2\sin\beta$, $\frac{g}{\omega^2} = r\cot\beta - \left(\frac{m_1 d_1{}^2 + m_2 d_2{}^2}{m_1 d_1 + m_2 d_2}\right)\cos\beta$ **M1A1**

That is $r\cot\beta = a + b\cos\beta$, where $b = \frac{m_1 d_1{}^2 + m_2 d_2{}^2}{m_1 d_1 + m_2 d_2}$ **A1 (10)**

12. (i) The probability distribution function of S_1 is

S_1	1	2	3	4	5	6
p	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

so the probability distribution function of R_1 is

R_1	0	1	2	3	4	5
p	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

and thus $G(x) = \frac{1}{6}(1 + t + t^2 + t^3 + t^4 + t^5)$. **B1**

The probability distribution function of S_2 is

S_2	2	3	4	5	6	7	8	9	10	11	12
p	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

M1

so the probability distribution function of R_2 is

R_2	0	1	2	3	4	5
p	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$

A1

which is the same as for R_1 and hence its probability generating function is also $G(x)$. **A1***

Therefore, the probability generating function of R_n is also $G(x)$ **B1**

and thus the probability that S_n is divisible by 6 is $1/6$. **B1 (6)**

(ii) The probability distribution function of T_1 is

T_1	0	1	2	3	4
p	$1/6$	$2/6$	$1/6$	$1/6$	$1/6$

and thus $G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4)$ **M1 A1**

$G_2(x)$ would be $\left(G_1(x)\right)^2$ except that the powers must be multiplied congruent to modulus 5.

$G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4) = \frac{1}{6}(x + 1 + x + x^2 + x^3 + x^4) = \frac{1}{6}(x + y)$ **B1**

Thus $G_2(x)$ would be $\frac{1}{36}(x + y)^2$

except $xy = x(1 + x + x^2 + x^3 + x^4) = x + x^2 + x^3 + x^4 + 1 = y$ **M1A1**

and $y^2 = (1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4) = (1 + x + x^2 + x^3 + x^4) + (x + x^2 + x^3 + x^4 + 1) + (x^2 + x^3 + x^4 + 1 + x) + (x^3 + x^4 + 1 + x + x^2) + (x^4 + 1 + x + x^2 + x^3) = 5y$ **A1**

So $G_2(x) = \frac{1}{36}(x + y)^2 = \frac{1}{36}(x^2 + 2xy + y^2) = \frac{1}{36}(x^2 + 2y + 5y) = \frac{1}{36}(x^2 + 7y)$ **M1A1* (8)**

$$G_3(x) = \frac{1}{6^3}(x + y)^3 = \frac{1}{6^3}(x + y)(x^2 + 7y) = \frac{1}{6^3}(x^3 + yx^2 + 7xy + 7y^2)$$

That is

$$G_3(x) = \frac{1}{6^3}(x^3 + yx^2 + 7xy + 7y^2) = \frac{1}{6^3}(x^3 + y + 7y + 35y) = \frac{1}{6^3}(x^3 + 43y)$$

We notice that the coefficient of y inside the bracket in $G_n(x)$ is $(1 + 6 + 6^2 + \cdots 6^{n-1})$

This can be shown simply by induction. It is true for $n = 1$ trivially.

Consider $(x + y)\left(x^r + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y\right) = x^{r+1} + yx^r + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)xy + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y^2$

$$\begin{aligned} yx^r + \left(1 + 6 + 6^2 + \cdots 6^{k-1}\right)xy + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y^2 \\ = y + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y + 5\left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y \end{aligned}$$

$$5\left(1 + 6 + 6^2 + \cdots 6^{k-1}\right) = (6 - 1)\left(1 + 6 + 6^2 + \cdots 6^{k-1}\right) = 6^k - 1$$

So $y + \left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y + 5\left(1 + 6 + 6^2 + \cdots + 6^{k-1}\right)y = \left(1 + 6 + 6^2 + \cdots 6^k\right)y$

as required. **M1**

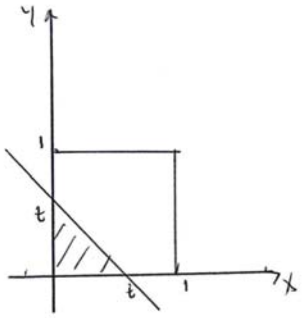
However, this coefficient is the sum of a GP and so $G_n(x) = \frac{1}{6^n}\left(x^{n-5p} + \frac{6^n - 1}{5}y\right)$ where p is an integer such that $0 \leq n - 5p \leq 4$. **M1 A1**

So if n is not divisible by 5, the probability that S_n is divisible by 5 will be the coefficient of x^0 which in turn is the coefficient of y, namely $\frac{1}{6^n}\left(\frac{6^n - 1}{5}\right) = \frac{1}{5}\left(1 - \frac{1}{6^n}\right)$ as required. **B1***

If n is divisible by 5, the probability that S_n is divisible by 5 will be $\frac{1}{6^n}\left(1 + \frac{6^n - 1}{5}\right)$ as $x^{n-5p} = x^0$

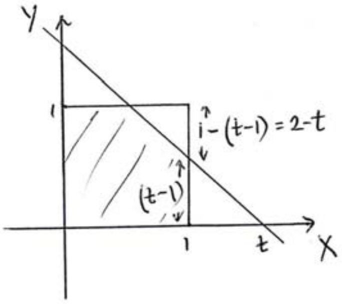
That is $\frac{1}{5}\left(1+\frac{4}{6^n}\right)$ **M1A1 (6)**

13. (i)



G1

$P(X+Y<t) = \frac{1}{2}t^2$ if $0 \le t \le 1$ **B1**



G1

and $P(X+Y<t) = 1-\frac{1}{2}(2-t)^2$ if $1 < t \le 2$ **B1**

$P(X+Y<t) = 0$ if $t<0$ and $P(X+Y<t) = 1$ if $t>2$

So $F(t) = \begin{cases} 0 \ \ for\ t<0 \\ \frac{1}{2}t^2 \ \ for\ 0 \le t \le 1 \\ 1-\frac{1}{2}(2-t)^2 \ \ for\ 1<t\le 2 \\ 1\ for\ t>2 \end{cases}$

B1 (5)

Thus $P((X+Y)^{-1}<t) = P\left(X+Y>\frac{1}{t}\right) = 1-P\left(X+Y<\frac{1}{t}\right)$

$$= \begin{cases} 1-\dfrac{1}{2t^2} \ \ for\ 1 \le t \\ \dfrac{1}{2}\left(2-\dfrac{1}{t}\right)^2 \ \ for\ \dfrac{1}{2} \le t < 1 \\ 0\ \ for\ t<\dfrac{1}{2} \end{cases}$$

M1 A1

So as $f(t) = \frac{dF(t)}{dt}$,

$$f(t)=\begin{cases} 0 \ \ for\ t<\frac{1}{2} \\ \frac{1}{t^2}\left(2-\frac{1}{t}\right) \ \ for\ \frac{1}{2}\le t<1 \\ \frac{1}{t^3} \ \ for\ 1\le t \end{cases}$$

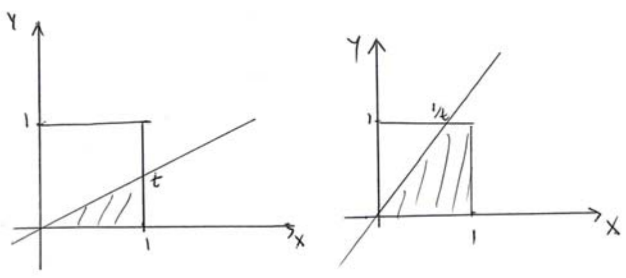
as required. **M1A1* (4)**

$$E\left(\frac{1}{X+Y}\right)=\int_{\frac{1}{2}}^{1} t(2t^{-2}-t^{-3})dt+\int_{1}^{\infty} t.t^{-3}dt=\ [2\ln t+t^{-1}]_{\frac{1}{2}}^{1}+[-t^{-1}]_{1}^{\infty}$$

$$=1-2\ln\frac{1}{2}-2+1=2\ln 2$$

M1 A1 (2)

(ii)



G1

$$P\left(\frac{Y}{X}<t\right)=\begin{cases} \frac{1}{2}t \ \ for\ 0\le t\le 1 \\ 1-\frac{1}{2}t^{-1} \ \ for\ t>1 \end{cases}$$

B1 (2)

Thus

$$P\left(\frac{X}{X+Y}<t\right)=P\left(\frac{X+Y}{X}>\frac{1}{t}\right)=P\left(1+\frac{Y}{X}>\frac{1}{t}\right)=P\left(\frac{Y}{X}>\frac{1}{t}-1\right)=1-P\left(\frac{Y}{X}<\frac{1}{t}-1\right)$$

So

$$F(t)=\begin{cases} 1-\frac{1}{2}\left(\frac{1}{t}-1\right) \ \ for\ \frac{1}{2}\le t\le 1 \\ \frac{1}{2}\left(\frac{1}{t}-1\right)^{-1} \ \ for\ 0\le t<\frac{1}{2} \end{cases}$$

i.e.

$$F(t)=\begin{cases} \frac{1}{2}(3-t^{-1}) \ \ for\ \frac{1}{2}\le t\le 1 \\ \frac{1}{2}\left(\frac{t}{1-t}\right) \ \ for\ 0\le t<\frac{1}{2} \end{cases}$$

M1A1

So as $f(t) = \frac{dF(t)}{dt}$,

$$f(t) = \begin{cases} \frac{1}{2}t^{-2} \quad for\ \frac{1}{2} \leq t \leq 1 \\ \frac{1}{2}(1-t)^{-2} \ \ for\ 0 \leq t < \frac{1}{2} \end{cases}$$

M1A1 (4)

$E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$ because, by symmetry, $E\left(\frac{X}{X+Y}\right) = E\left(\frac{Y}{X+Y}\right)$

and $E\left(\frac{X}{X+Y}\right) + E\left(\frac{Y}{X+Y}\right) = E\left(\frac{X+Y}{X+Y}\right) = E(1) = 1$ **B1**

$$E\left(\frac{X}{X+Y}\right) = \int_0^{\frac{1}{2}} \frac{1}{2}t(1-t)^{-2}\,dt + \int_{\frac{1}{2}}^{1} t \times \frac{1}{2}t^{-2}\,dt$$

$$= \left[\frac{1}{2}t(1-t)^{-1}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{1}{2}(1-t)^{-1}dt + \left[\frac{1}{2}\ln t\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{2} - \left[-\frac{1}{2}\ln(1-t)\right]_0^{\frac{1}{2}} - \frac{1}{2}\ln\frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}$$

as required. **M1A1 (3)**

The Admissions Testing Service is part of Cambridge English, a not-for-profit department of the University of Cambridge. It offers a range of tests and tailored assessment services to support selection and recruitment for governments, educational institutions and professional organisations around the world. Underpinned by robust and rigorous research, its services include:

- assessments in thinking skills
- admission tests for medicine and healthcare
- behavioural styles assessment
- subject-specific aptitude tests.

The Admissions Testing Service
University of Cambridge ESOL Examinations
1 Hills Road
Cambridge CB1 2EU
United Kingdom
Tel: +44 (0)1223 553366
Email: admissionstests@cambridgeassessment.org.uk

UNIVERSITY of CAMBRIDGE
ESOL Examinations