STEP 2014, P3, Q13 - Sol'n (2 pages; 3/6/20)

(i) [By "pgf conditional on this happening" is presumably meant "pgf of the (random variable representing the) score, given that the game ends in the 1st round"]

Required pgf ($G_1(t)$, say) is:

$P(score\ is$ 0 | game ends in 1st round) . t^0

$+ P(score\ is$ 1 | game ends in 1st round) . t^1

$+ P(score\ is$ 2 | game ends in 1st round) . t^2

$+$ ...

$= (1)(1) + (0)t + (0)t^2$ ...

$= 1$, as required.

(ii) Required pgf ($G_2(t)$, say) is:

$P(score\ is$ 0 | game continues to next round with no change in score) . t^0

$+ P(score\ is$ 1 | game continues to next round with no change in score) . t^1

$+ P(score\ is$ 2 | game continues to next round with no change in score) . t^2

$+$ ...

Let N' be the final score, starting from round 2.

Then required pgf is

$P(N' = 0) + P(N' = 1)t + P(N' = 2)t^2 + \cdots$

$= G(t)$, as N & N' have the same probability distribution.

(iii) **1st part**

pgf given that the game continues to the 2nd round, with the score increased by 1 ($G_3(t)$, say) is

$0 + P(N' = 0)t + P(N' = 1)t^2 + \cdots$

$= t(P(N' = 0) + P(N' = 1)t + P(N' = 2)t^2 + \cdots)$

$= tG(t)$

Then $G(t) = aG_1(t) + bG_2(t) + cG_3(t)$

[as the coeff. of t^n is $P(N = n)$, and this can be determined by conditioning on what happens in the 1st round]

$= a + bG(t) + ctG(t)$, as required.

2nd part

So $G(t)\{1 - b - ct) = a$, and $G(t) = a(1 - b - ct)^{-1}$

$= \frac{a}{1-b}(1 - \frac{ct}{1-b})^{-1}$

$= \frac{a}{1-b}(1 + \frac{ct}{1-b} + \left(\frac{ct}{1-b}\right)^2 + \cdots)$

and $P(N = n) =$ coeff. of $t^n = \frac{a}{1-b}\left(\frac{c}{1-b}\right)^n = \frac{ac^n}{(1-b)^{n+1}}$, as required.

(iv) $E(N) = G'(1)$

$G(t) = a(1 - b - ct)^{-1}$

$\Rightarrow G'(t) = -a(1 - b - ct)^{-2}(-c) = ac(1 - b - ct)^{-2}$

and so $G'(1) = ac(1 - b - c)^{-2} = ac.a^{-2} = \frac{c}{a}$

Thus $\mu = \frac{c}{a}$, and $P(N = n) = \frac{ac^n}{(1-b)^{n+1}} = \frac{\left(\frac{c}{a}\right)^n}{\left(\frac{a+c}{a}\right)^{n+1}} = \frac{\mu^n}{(1+\mu)^{n+1}}$, as required.