

STEP 2014, P2, Q5 - Solution (2 pages; 30/6/20)

$$y = xu \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$(i) \text{ With } y = xu, \frac{dy}{dx} = \frac{2y+x}{y-2x} \Rightarrow u + x \frac{du}{dx} = \frac{2xu+x}{xu-2x} = \frac{2u+1}{u-2}$$

$$\Rightarrow x \frac{du}{dx} = \frac{2u+1}{u-2} - u = \frac{2u+1-u(u-2)}{u-2} = \frac{4u+1-u^2}{u-2}$$

$$\Rightarrow \int \frac{u-2}{4u+1-u^2} du = \int \frac{1}{x} dx$$

$$\Rightarrow \ln|x| = -\frac{1}{2} \int \frac{-2u+4}{4u+1-u^2} du = -\frac{1}{2} \ln|4u+1-u^2| - \ln C$$

[without loss of generality, as $\ln C$ can take any value]

$$\Rightarrow C|x| = |4u+1-u^2|^{-\frac{1}{2}}$$

$$\Rightarrow C^2 x^2 = \frac{1}{|4u+1-u^2|} \quad (A)$$

$$\text{As sol'n passes through } (1,1), C^2 = \frac{1}{|4+1-1|} = \frac{1}{4}$$

So $x^2(4u+1-u^2) = 4$ (in order for the curve to pass through (1,1))

$$\text{and hence } x^2\left(\frac{4y}{x} + 1 - \frac{y^2}{x^2}\right) = 4$$

$$\Rightarrow 4xy + x^2 - y^2 = 4$$

$$\text{or } x^2 - y^2 + 4xy - 4 = 0$$

(ii) Let $x = X + a$ & $y = Y + b$,

$$\text{so that } \frac{x-2y-4}{2x+y-3} = \frac{X+a-2(Y+b)-4}{2(X+a)+Y+b-3}$$

$$\text{Then set } a - 2b - 4 = 0 \text{ \& } 2a + b - 3 = 0,$$

$$\text{so that } 2(2b+4) + b - 3 = 0; 5b = -5; b = -1; a = 2$$

$$\text{Then } \frac{dy}{dx} = \frac{x-2y-4}{2x+y-3} \Rightarrow \frac{dY}{dX} = \frac{X-2Y}{2X+Y} \text{ or } \frac{dX}{dY} = \frac{2X+Y}{X-2Y},$$

which is the differential eq'n in (i), with X in place of y and Y in place of x .

$$\text{So, from (A) in (i), } C^2 Y^2 = \frac{1}{|4U+1-U^2|}, \text{ where } X = YU$$

$$\text{When } x = 1, X = 1 - 2 = -1,$$

$$\text{and when } y = 1, Y = 1 - (-1) = 2$$

$$\text{Hence } C^2(4) = \frac{1}{|4(-\frac{1}{2})+1-(-\frac{1}{2})^2|} = \frac{1}{|-2+1-\frac{1}{4}|} = \frac{1}{(\frac{5}{4})}$$

$$\text{and so } C^2 = \frac{1}{5}, \text{ giving } Y^2 = \frac{5}{-(4U+1-U^2)}$$

(in order for the curve to pass through (1, 1))

$$\text{and so } (y+1)^2 = \frac{5}{-(4(\frac{x-2}{y+1})+1-(\frac{x-2}{y+1})^2)}$$

$$= \frac{5(y+1)^2}{-4(x-2)(y+1)-(y+1)^2+(x-2)^2}$$

$$\Rightarrow (y+1)^2(-4xy - 4x + 8y + 8 - y^2 - 2y - 1 + x^2 - 4x + 4)$$

$$= 5(y+1)^2$$

$$\Rightarrow (y+1)^2(-4xy - 8x + 6y + 6 - y^2 + x^2) = 0$$

$$\text{or } (y+1)^2(x^2 - y^2 - 8x + 6y - 4xy + 6) = 0$$