STEP 2014, P2, Q4 - Sol'n (2 pages; 24/6/20)

(i) $u = \frac{1}{x} \Rightarrow du = -x^{-2}dx \Rightarrow dx = -x^2 du = -u^{-2}du$

Then $I = \int_{\frac{1}{b}}^{b} \frac{x\ln x}{(a^2+x^2)(a^2x^2+1)} dx = \int_{b}^{\frac{1}{b}} \frac{\frac{1}{u}\ln(\frac{1}{u})(-u^{-2})}{(a^2+u^{-2})(a^2u^{-2}+1)}\, du$

$= \int_{\frac{1}{b}}^{b} \frac{-u\ln u}{(a^2u^2+1)(a^2+u^2)}\, du = -I$, so that $I = 0$, as required.

(ii) With $u = \frac{1}{x}$, $J = \int_{\frac{1}{b}}^{b} \frac{\arctan x}{x}\, dx = \int_{b}^{\frac{1}{b}} \frac{\arctan\left(\frac{1}{u}\right)(-u^{-2})du}{(\frac{1}{u})}$

$= \int_{\frac{1}{b}}^{b} \frac{\left(\frac{\pi}{2} - \arctan u\right)}{u}\, du$

$= \frac{\pi}{2}\int_{\frac{1}{b}}^{b} \frac{1}{u} du - J,$

so that $2J = \frac{\pi}{2}(\ln b - \ln\left(\frac{1}{b}\right))$ (as $b > 0$),

and $J = \frac{\pi}{4}(2\ln b) = \frac{\pi \ln b}{2}$, as required.

(iii) Let $u = \frac{k}{x}$, so that $du = -kx^{-2}dx$, and $dx = -\frac{k^2u^{-2}}{k} du$

Then $K = \int_0^{\infty} \frac{1}{(a^2+x^2)^2} dx = \int_{\infty}^{0} \frac{1}{(a^2+k^2u^{-2})^2}(-ku^{-2})du$

$= k\int_0^{\infty} \frac{u^2}{(a^2u^2+k^2)^2}\, du$

[Aiming to involve K again] Let $k = a^2$, so that

$K = a^{-2}\int_0^{\infty} \frac{u^2}{(u^2+a^2)^2}\, du$

$= a^{-2}\int_0^{\infty} \frac{u^2+a^2}{(u^2+a^2)^2}\, du - a^{-2}\int_0^{\infty} \frac{a^2}{(u^2+a^2)^2}\, du$

$= a^{-2} \int_0^{\infty} \frac{1}{u^2+a^2}\, du - K$

Hence $2K = a^{-2}\left(\frac{\pi}{2a}\right)$, using the given result (for $a > 0$)

that $\int_0^{\infty} \frac{1}{a^2+x^2}\, dx = \frac{\pi}{2a}$

And so $K = \frac{\pi}{4a^3}$ (for $a > 0$), as required.