STEP 2013, P3, Q8 - Solution (2 pages; 9/7/20)

1st part

$\sum_{r=0}^{n-1} e^{2i(\alpha+\frac{r\pi}{n})} = e^{2i\alpha}(1 + e^{\frac{2i\pi}{n}} + \cdots + e^{\frac{2i(n-1)\pi}{n}})$

$= e^{2i\alpha}\frac{\{\left(e^{\frac{2i\pi}{n}}\right)^n - 1\}}{e^{\frac{2i\pi}{n}} - 1}$

$= e^{2i\alpha}\frac{(e^{2i\pi}-1)}{e^{\frac{2i\pi}{n}}-1} = 0$, as $e^{2i\pi} = 0$ and $e^{\frac{2i\pi}{n}} \neq 1$

2nd part

$s = d - rcos\theta$

3rd part

Without loss of generality, we can assume that L_1 is horizontal, L_2 is at an angle $\frac{\pi}{n}$ to (and above) the horizontal, etc

$l_j = r(\theta_j) + r(-[\pi - \theta_j])$

[We need to express $r(\theta)$ in terms of d]

$s = d - rcos\theta$ and $r = ks$,

so that $\frac{r}{k} = d - rcos\theta$

$\Rightarrow r\left(\frac{1}{k} + cos\theta\right) = d$

$\Rightarrow r = \frac{d}{\frac{1}{k}+cos\theta}$

and $l_j = \frac{d}{\frac{1}{k}+cos\theta_j} + \frac{d}{\frac{1}{k}+\cos(-[\pi-\theta_j])}$

$= \frac{d}{\frac{1}{k}+cos\theta_j} + \frac{d}{\frac{1}{k}+\cos(\pi-\theta_j)}$

$= \frac{d}{\frac{1}{k}+cos\theta_j} + \frac{d}{\frac{1}{k}-cos\theta_j}$

$= \frac{d\left(\frac{1}{k}-cos\theta_j\right)+d(\frac{1}{k}+cos\theta_j)}{\frac{1}{k^2}-cos^2\theta_j}$

$= \frac{(\frac{2d}{k})}{\frac{1}{k^2}-cos^2\theta_j}$

and $\frac{1}{l_j} = \frac{1-k^2cos^2\theta_j}{2kd}$

Then $\sum_{j=1}^{n}\frac{1}{l_j} = \frac{n}{2kd} - \frac{k^2}{4kd}\sum_{j=1}^{n}2cos^2\theta_j$ (1)

Result to prove: $\sum_{j=1}^{n}2cos^2\theta_j = n$

Proof: LHS $= \sum_{j=1}^{n}\{1+\cos(2\theta_j)\} = n + \sum_{j=1}^{n}\cos(2\theta_j)$ (2)

The initial result, $\sum_{r=0}^{n-1}e^{2i(\alpha+\frac{r\pi}{n})} = 0$

$\Rightarrow Re\{\sum_{r=0}^{n-1}e^{2i(\alpha+\frac{r\pi}{n})}\} = 0$

$\Rightarrow \sum_{r=0}^{n-1}\cos\left(2\alpha+\frac{2r\pi}{n}\right) = 0$

Setting $\alpha = 0$, and as $\theta_j = \frac{(j-1)\pi}{n}$,

it follows that $\sum_{j=1}^{n}\cos(2\theta_j) = 0$

Thus $(2) = n$, as required.

And so $(1) = \frac{n}{2kd} - \frac{k^2n}{4kd} = \frac{(2-k^2)n}{4kd}$, as required.