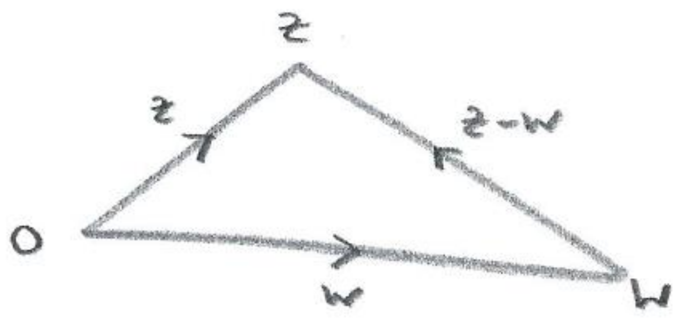
STEP 2013, P3, Q6 - Solution (3 pages; 20/7/20)



Referring to the diagram, $|z-w| = WZ$, which is no longer than $WO + OZ = |w| + |z|$; ie $|z-w| \leq |z| + |w|$

(i) **1st part**

$|z-w|^2 = (z-w)(z-w)^*$

$= (z-w)(z^* - w^*)$

$= zz^* - zw^* - wz^* + ww^*$

$= |z|^2 - (E - 2|zw|) + |w|^2$

$= |z|^2 + 2|zw| + |w|^2 - E$

$= (|z| + |w|)^2 - E$, as required.

2nd part

As $|z|, |w|$ & $|z-w|$ are all real,

$E = (|z| + |w|)^2 - |z-w|^2$ is real, as required.

3rd part

From the initial result, $|z-w| \leq |z| + |w|$

$\Rightarrow |z-w|^2 \leq (|z|+|w|)^2$

$\Rightarrow E = (|z|+|w|)^2 - |z-w|^2 \geq 0$, as required.

(ii) **1st part**

Equivalent result to prove: $(1+|zw|)^2 - |1-zw^*|^2 = E$

LHS $= 1 + 2|zw| + |zw|^2 - (1-zw^*)(1-zw^*)^*$

$= 1 + 2|zw| + |z|^2|w|^2 - (1-zw^*)(1-z^*w)$

$= 1 + 2|zw| + |z|^2|w|^2 - (1 - z^*w - zw^* + zw^*z^*w)$

$= 2|zw| + z^*w + zw^*$ (as $zw^*z^*w = zz^*ww^* = |z|^2|w|^2$)

$= E$, as required.

2nd part

From (i) & (ii),

$|z-w|^2 = (|z|+|w|)^2 - E$ and $|1-zw^*|^2 = (1+|zw|)^2 - E$,

and hence $\frac{|z-w|^2}{|1-zw^*|^2} = \frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E}$ (A)

Result to prove: For $a, b, c, a-c, b-c > 0$, $\frac{a-c}{b-c} < \frac{a}{b}$ (B), under certain circumstances.

Proof

(B) $\Leftrightarrow ab - cb < ab - ac$, as $b > 0$ & $b - c > 0$

$\Leftrightarrow b > a$, as $c > 0$

So $\frac{a-c}{b-c} < \frac{a}{b}$ when $a, b, c, a-c, b-c, b-a > 0$

Then, applying the above result to (A):

$\frac{|z-w|^2}{|1-zw^*|^2} = \frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E} < \frac{(|z|+|w|)^2}{(1+|zw|)^2}$ (C), if $E > 0$, provided that

$(|z| + |w|)^2 < (1 + |zw|)^2$ (D)

(as $|z - w|^2, |1 - zw^*|^2, (|z| + |w|)^2 - E$ & $(1 + |zw|)^2 - E > 0$)

and (D) will be true if $|z| + |w| < 1 + |zw|$ (F)

(F) $\Leftrightarrow 1 + |z||w| - |z| - |w| > 0$

$\Leftrightarrow |z|(|w| - 1) + (1 - |w|) > 0$

$\Leftrightarrow (1 - |z|)(1 - |w|) > 0$ (G)

and (G) holds, as $1 - |z| < 0$ & $1 - |w| < 0$

So (C) holds if $E > 0$, and if $E = 0$, $\frac{|z-w|^2}{|1-zw^*|^2} = \frac{(|z|+|w|)^2}{(1+|zw|)^2}$

Hence, $\frac{|z-w|^2}{|1-zw^*|^2} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2}$ (H)

3rd part

If both $|z| < 1$ & $|w| < 1$, then (G) also holds, and hence (H) does.