STEP 2013, P3, Q1 - Solution (3 pages; 6/4/21)

1st part

$t = \tan\left(\frac{1}{2}x\right) \Rightarrow \frac{dt}{dx} = sec^2\left(\frac{1}{2}x\right).\frac{1}{2} = \frac{1}{2}(1 + tan^2\left(\frac{1}{2}x\right))$

$= \frac{1}{2}(1 + t^2)$

2nd part

$sinx = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2\tan\left(\frac{x}{2}\right)}{sec^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$

3rd part

Write $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+asinx}\, dx$, and let $t = \tan\left(\frac{1}{2}x\right)$

Then $\frac{dt}{dx} = \frac{1}{2}(1 + t^2)$ and hence $dx = \frac{2}{1+t^2}\, dt$

And $I = \int_0^1 \frac{1}{1+a\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt = 2\int_0^1 \frac{1}{1+t^2+2at}\, dt$

Now $1 + t^2 + 2at = (t + a)^2 - a^2 + 1$,

so that $I = 2\left[\frac{1}{\sqrt{1-a^2}} arctan\left(\frac{t+a}{\sqrt{1-a^2}}\right)\right]\begin{matrix}1\\0\end{matrix}$

$= \frac{2}{\sqrt{1-a^2}}\{\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right)\}$

Now, if $tan\theta = t$ & $tan\phi = T$ (where $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$),

then $\tan(\theta - \phi) = \frac{tan\theta - tan\phi}{1+tan\theta tan\phi}$

$= \frac{t-T}{1+tT}$, and $\arctan(t) - \arctan(T) = \theta - \phi = \arctan\left(\frac{t-T}{1+tT}\right)$

Thus $\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right) = \arctan\left(\frac{\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\left(\frac{a}{\sqrt{1-a^2}}\right)}{1+\left(\frac{1+a}{\sqrt{1-a^2}}\right)\left(\frac{a}{\sqrt{1-a^2}}\right)}\right)$

$= \arctan\left(\frac{\sqrt{1-a^2}}{1-a^2+(a+a^2)}\right) = \arctan\left(\frac{\sqrt{1-a^2}}{1+a}\right) = \arctan\left(\sqrt{\frac{(1-a)(1+a)}{(1+a)^2}}\right)$

$= \arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)$

and so $I = \frac{2}{\sqrt{1-a^2}} \arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)$, as required.

4th part

If $I_n = \int_0^{\frac{\pi}{2}} \frac{sin^n x}{2+sinx}\, dx$, then $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \frac{sin^{n+1}x+2sin^n x}{2+sinx}\, dx$

$= \int_0^{\frac{\pi}{2}} sin^n x\, dx$

Hence $I_3 + 2I_2 = \int_0^{\frac{\pi}{2}} sin^2 x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - cos2x)\, dx$

$= \frac{1}{2}\left[x - \frac{1}{2}sin2x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{2}(0 - 0) = \frac{\pi}{4}$ (*)

And $I_2 = \int_0^{\frac{\pi}{2}} \frac{sin^2 x}{2+sinx}\, dx = \int_0^{\frac{\pi}{2}} \frac{sin^2 x+2sinx}{2+sinx} - \frac{2sinx}{2+sinx}\, dx$

$= \int_0^{\frac{\pi}{2}} sinx - \frac{2sinx+4}{2+sinx} + \frac{4}{2+sinx}\, dx$

$= [-cosx]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\, dx + 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{2}sinx}\, dx$

$= (0 + 1) - 2\left(\frac{\pi}{2} - 0\right) + \frac{4}{\sqrt{1-\frac{1}{4}}} \arctan\left(\frac{\sqrt{1-\frac{1}{2}}}{\sqrt{1+\frac{1}{2}}}\right)$, from the 3rd part

$= 1 - \pi + \frac{8}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right) = 1 - \pi + \frac{8}{\sqrt{3}}\left(\frac{\pi}{6}\right)$

So, from (*), $I_3 = \frac{\pi}{4} - 2I_2 = \frac{\pi}{4} - 2(1 - \pi + \frac{8}{\sqrt{3}}(\frac{\pi}{6}))$

$= \pi\left(\frac{9}{4} - \frac{8}{3\sqrt{3}}\right) - 2$ or $\left(\frac{9}{4} - \frac{8\sqrt{3}}{9}\right)\pi - 2$