STEP 2013, Paper 2, Q13 – Sol'n (3 pages; 11/6/20)

(i) **1st part**

$P(A = 1) = P(\text{1st throw is H})P(\text{2nd throw is H})$

$+P(\text{1st throw is T})P(\text{2nd throw is T})$

$= p^2 + q^2$

2nd part

$P(S = 1) = P(\text{1st throw is H})P(\text{2nd throw is T})$

$+P(\text{1st throw is T})P(\text{2nd throw is H})$

$= 2pq$

3rd part

$P(A = 1) - P(S = 1) = p^2 + q^2 - 2pq = (p - q)^2 > 0$, as $p \neq q$

So $P(S = 1) < P(A = 1)$, as required.

(ii) **1st part**

$P(S = 2) = P(HHT) + P(TTH)$

$= p^2q + q^2p = pq(p + q) = pq$

And $P(A = 2) = P(HTT) + P(THH)$

$= pq^2 + qp^2 = pq(q + p) = pq$

Thus $P(S = 2) = P(A = 2)$, as required.

2nd part

$P(S = 3) = P(HHHT) + P(TTTH)$

$= p^3q + q^3p = pq(p^2 + q^2)$

And $P(A = 3) = P(HTHH) + P(THTT)$

$= pqp^2 + qpq^2 = pq(p^2 + q^2)$

Thus $P(S = 3) = P(A = 3)$.

(iii) **1st part**

$P(S = 2n) = P(HHHH \dots [2n\ times]T)$

$+P(TTTT \dots [2n\ times]H)$

$= p^{2n}q + q^{2n}p$

And $P(A = 2n) = P(HTHT \dots [2n\ items]T)$

$+P(THTH \dots [2n\ items]H)$

$= (pq)^n q + (qp)^n p$

Then $P(S = 2n) - P(A = 2n)$

$= p^{2n}q + q^{2n}p - (pq)^n q - (qp)^n p$

$= p^n q(p^n - q^n) + q^n p(q^n - p^n)$

$= (p^n - q^n)(p^n q - q^n p)$

$= (p^n - q^n)pq(p^{n-1} - q^{n-1})$ (A)

If $n = 2$, (A) $= (p^2 - q^2)pq(p - q)$

$= (p - q)^2(p + q)pq > 0$, as $(p - q)^2 > 0$ (as $p \neq q$)

If $n > 2$, (A) $= (p - q)(p^{n-1} + qp^{n-2} + \dots + q^{n-1})pq$

$. (p - q)(p^{n-2} + qp^{n-3} + \dots + q^{n-2}) > 0$, similarly.

so that $P(S = 2n) > P(A = 2n)$ (for $n > 1$), as required.

2nd part

$P(S = 2n + 1) = P(HHHH \dots [2n + 1\ times]T)$

$+P(TTTT \dots [2n + 1\ times]H)$

$= p^{2n+1}q + q^{2n+1}p$

And $P(A = 2n + 1) = P(HTHT \dots H[2n + 1\ items]H)$

$+P(THTH \dots T[2n + 1\ items]T)$

$= (pq)^n p^2 + (qp)^n q^2$

Then $P(S = 2n + 1) - P(A = 2n + 1)$

$= p^{2n+1}q + q^{2n+1}p - (pq)^n p^2 - (qp)^n q^2$

$= p^{n+2}q(p^{n-1} - q^{n-1}) + q^{n+2}p(q^{n-1} - p^{n-1})$

$= pq(p^{n+1} - q^{n+1})(p^{n-1} - q^{n-1}) > 0$, by the same reasoning.

Thus, $P(S = 2n + 1) > P(A = 2n + 1)$ for $n > 1$.