STEP 2013, Paper 2, Q12 – Sol'n (2 pages; 7/4/21)

[This was apparently the least attempted question on the paper, but is relatively straightforward and requires only basic knowledge of the Poisson distribution and familiarity with the $E[X(X-1)]$ device for determining $Var(X)$.]

(i) $E(X) = 1.\frac{e^{-\lambda}\lambda}{1!} + 3.\frac{e^{-\lambda}\lambda^3}{3!} + 5.\frac{e^{-\lambda}\lambda^5}{5!} + \cdots$

$= e^{-\lambda}\lambda(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots)$

$= e^{-\lambda}\lambda\alpha$

And $E(Y) = 2.\frac{e^{-\lambda}\lambda^2}{2!} + 4.\frac{e^{-\lambda}\lambda^4}{4!} + \cdots$

$= e^{-\lambda}\lambda(\frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots)$

$= e^{-\lambda}\lambda\beta$

[Also, $E(X+Y) = E(U) = \lambda$, and

$E(X) + E(Y) = e^{-\lambda}\lambda\alpha + e^{-\lambda}\lambda\beta = e^{-\lambda}\lambda(\alpha+\beta) = e^{-\lambda}\lambda e^{\lambda} = \lambda$]

(ii) **1st part**

$E[X(X-1)] = 3(2)\frac{e^{-\lambda}\lambda^3}{3!} + 5(4)\frac{e^{-\lambda}\lambda^5}{5!} + \cdots$

$= e^{-\lambda}\lambda^2(\frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots) = e^{-\lambda}\lambda^2\beta$

Then $Var(X) = E[X(X-1)] + E(X) - [E(X)]^2$

$= e^{-\lambda}\lambda^2\beta + e^{-\lambda}\lambda\alpha - (e^{-\lambda}\lambda\alpha)^2$

Then, as $\alpha + \beta = e^{\lambda}$,

$Var(X) = \frac{\lambda^2\beta+\lambda\alpha}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$, as required.

Also, $E[Y(Y-1)] = 2(1)\frac{e^{-\lambda}\lambda^2}{2!} + 4(3)\frac{e^{-\lambda}\lambda^4}{4!} + \cdots$

$= e^{-\lambda}\lambda^2(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots) = e^{-\lambda}\lambda^2\alpha$; ie as for X, but with β replaced by α.

and so $Var(Y) = \frac{\lambda^2\alpha+\lambda\beta}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2}$ (swapping α & β in $Var(X)$)

2nd part

$Var(X) + Var(Y) = Var(X+Y)$

$\Leftrightarrow \frac{\lambda^2\beta+\lambda\alpha}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} + \frac{\lambda^2\alpha+\lambda\beta}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} = Var(U) = \lambda$

$\Leftrightarrow (\lambda^2\beta + \lambda\alpha)(\alpha+\beta) - \lambda^2\alpha^2 + (\lambda^2\alpha + \lambda\beta)(\alpha+\beta) - \lambda^2\beta^2$

$= \lambda e^{2\lambda}$ (as $\alpha + \beta = e^{\lambda}$)

$\Leftrightarrow (\lambda^2\beta + \lambda\alpha)(\alpha+\beta) - \lambda^2\alpha^2 + (\lambda^2\alpha + \lambda\beta)(\alpha+\beta) - \lambda^2\beta^2$

$= \lambda e^{2\lambda}$

$\Leftrightarrow (\lambda\beta + \alpha)(\alpha+\beta) - \lambda\alpha^2 + (\lambda\alpha + \beta)(\alpha+\beta) - \lambda\beta^2 = e^{2\lambda}$

$\Leftrightarrow (\lambda\beta + \alpha + \lambda\alpha + \beta)(\alpha+\beta) - \lambda(\alpha^2 + \beta^2) = e^{2\lambda}$

$\Leftrightarrow (\lambda+1)(\alpha+\beta)(\alpha+\beta) - \lambda(\alpha^2+\beta^2) = (\alpha+\beta)^2$

$\Leftrightarrow \lambda(\alpha+\beta)^2 = \lambda(\alpha^2+\beta^2)$

Thus, if $\lambda \neq 0$, $(\alpha+\beta)^2 = \alpha^2 + \beta^2 \Leftrightarrow \alpha\beta = 0$

So, as neither α nor β is zero, there are no non-zero values of λ for which $Var(X) + Var(Y) = Var(X+Y)$.