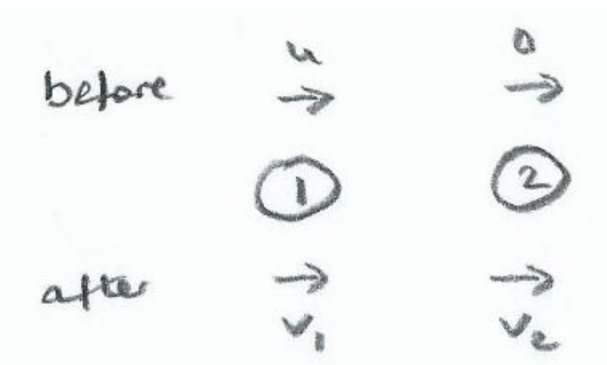
STEP 2013, Paper 2, Q11 – Sol'n (4 pages; 7/6/20)

(i) 1st part

1st collision (between the 1st and 2nd particles, counting from the left, assuming that the 1st particle is initially moving to the right):



CoM: $mu + 0 = mv_1 + mv_2 \Rightarrow u = v_1 + v_2$ (1)

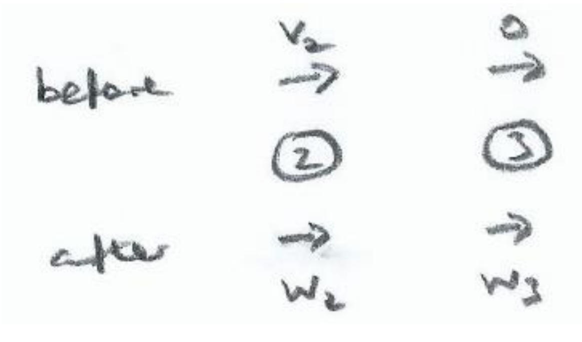
NLR: $v_2 - v_1 = eu$ (2)

$(1) + (2) \Rightarrow 2v_2 = (1 + e)u$

$(1) - (2) \Rightarrow 2v_1 = (1 - e)u$

So $v_1 = \frac{1}{2}(1 - e)u$ and $v_2 = \frac{1}{2}(1 + e)u$

2nd collision (between the 2nd and 3rd particles):



CoM: $mv_2 + 0 = mw_2 + mw_3 \Rightarrow v_2 = w_2 + w_3$ (3)

NLR: $w_3 - w_2 = ev_2$ (4)

$(3) + (4) \Rightarrow 2w_3 = (1+e)v_2$

$(3) - (4) \Rightarrow 2w_2 = (1-e)v_2$

So $w_2 = \frac{1}{2}(1-e).\frac{1}{2}(1+e)u = \frac{1}{4}u(1-e^2)$

and $w_3 = \frac{1}{2}(1+e).\frac{1}{2}(1+e)u = \frac{1}{4}u(1+e)^2$

2nd part

There will be a 3rd collision (between the 1st and 2nd particles) if $v_1 > w_2$.

$v_1 - w_2 = \frac{1}{2}(1-e)u - \frac{1}{4}u(1-e^2)$

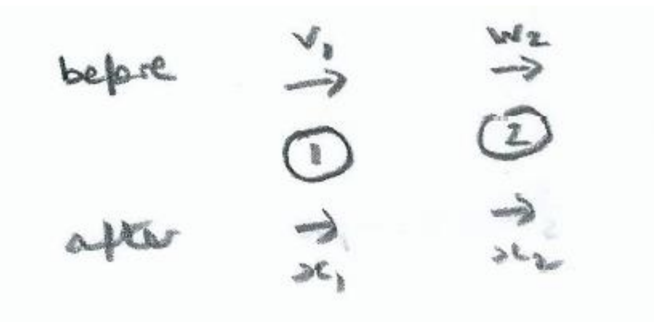
$= \frac{1}{4}(1-e)u(2-[1+e])$

$= \frac{1}{4}(1-e)u(1-e)$

$= \frac{1}{4}u(1-e)^2 > 0$, as $e < 1$

Thus $v_1 > w_2$ and there will be a 3rd collision.

(ii) If there is a 4th collision it will occur between the 2nd and 3rd particles.



A 4th collision will occur if $x_2 > w_3$.

CoM: $mv_1 + mw_2 = mx_1 + mx_2 \Rightarrow v_1 + w_2 = x_1 + x_2$

$\Rightarrow \frac{1}{2}(1-e)u + \frac{1}{4}u(1-e^2) = x_1 + x_2 \quad (5)$

NLR: $x_2 - x_1 = e(v_1 - w_2)$

$\Rightarrow x_2 - x_1 = e\{\frac{1}{2}(1-e)u - \frac{1}{4}u(1-e^2)\} \quad (6)$

$(5) + (6) \Rightarrow$

$2x_2 = \frac{1}{2}(1-e)u + \frac{1}{4}u(1-e^2) + e\{\frac{1}{2}(1-e)u - \frac{1}{4}u(1-e^2)\}$

$= \frac{1}{4}u\{2(1-e) + (1-e^2) + 2e(1-e) - e(1-e^2)\}$

From the 1st part of (i), $w_3 = \frac{1}{4}u(1+e)^2$,

so $x_2 > w_3$ when

$\frac{1}{2}\{2(1-e) + (1-e^2) + 2e(1-e) - e(1-e^2)\} > (1+e)^2$

ie when

$2(1-e) + (1-e^2) + 2e(1-e) - e(1-e^2) - 2(1+e)^2 > 0 \quad$ (A)

LHS of (A)

$= e^3 + e^2(-1-2-2) + e(-2+2-1-4) + (2+1-2)$

$= e^3 - 5e^2 - 5e + 1$

Let $f(e) = e^3 - 5e^2 - 5e + 1$

[Note that $f(0) > 0$ & $f(1) < 0$, so that $f(e) = 0$ for some e between 0 & 1. Also, it can be shown that any rational solution of $f(e) = 0$ is an integer (as the coeff. of e^3 is 1).]

As $f(-1) = -1 - 5 + 5 + 1 = 0$,

$f(e) = (e+1)(e^2 - 6e + 1)$

and then $e^2 - 6e + 1 = 0 \Rightarrow e = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

Thus the root of $f(e) = 0$ lying between 0 & 1 is $3 - 2\sqrt{2}$,

and so $x_2 > w_3$ when $f(e) > 0$, which occurs when $e < 3 - 2\sqrt{2}$