STEP 2013, P1, Q1 - Sol'n (2 pages; 19/6/20)

(i) $y^2 + 3y - \frac{1}{2} = 0 \Rightarrow 2y^2 + 6y - 1 = 0$

$\Rightarrow y = \frac{-6\pm\sqrt{36+8}}{4} = \frac{-3+\sqrt{11}}{2}$ (rejecting -ve root, as $y = \sqrt{x} > 0$)

So $x = \frac{(-3+\sqrt{11})^2}{4} = \frac{1}{4}\left(9 + 11 - 6\sqrt{11}\right) = 5 - \frac{3}{2}\sqrt{11}$

(ii)(a) Let $y = \sqrt{x+2}$, so that $(y^2 - 2) + 10y - 22 = 0$

$\Rightarrow y^2 + 10y - 24 = 0$

$\Rightarrow y = \frac{-10\pm\sqrt{100+96}}{2} = -5 + \frac{14}{2} = 2$

(rejecting -ve root, as $y = \sqrt{x+2} > 0$)

$\Rightarrow x = y^2 - 2 = 2$

(b) Let $y = \sqrt{2x^2 - 8x - 3}$, so that $2x^2 - 8x - 3 = y^2$ (A)

$\Rightarrow x^2 - 4x = \frac{1}{2}(y^2 + 3)$ [fortunately]

So the eq'n in the question becomes $\frac{1}{2}(y^2 + 3) + y - 9 = 0$

$\Rightarrow y^2 + 2y - 15 = 0$

$\Rightarrow (y + 5)(y - 3) = 0$

$\Rightarrow y = 3$ (as $y > 0$)

$\Rightarrow 2x^2 - 8x - 3 = 9$, from (A)

$\Rightarrow 2x^2 - 8x - 12 = 0$

$\Rightarrow x^2 - 4x - 6 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16+24}}{2} = 2 \pm \sqrt{10}$

Checking for spurious sol'ns:

$x^2 = 4 + 10 \pm 4\sqrt{10}$

So, if $x = 2 + \sqrt{10}$, LHS of the eq'n in the question is

$14 + 4\sqrt{10} - 4\left(2 + \sqrt{10}\right) + 3 - 9$, which equals 0

If $x = 2 - \sqrt{10}$, LHS of the eq'n in the question is

$14 - 4\sqrt{10} - 4\left(2 - \sqrt{10}\right) + 3 - 9$, which also equals 0

So $x = 2 \pm \sqrt{10}$ are the real sol'ns.

[The Examiner's Report says that it is 'very easy' to explain (without direct verification) that the two roots are correct, but I'm not sure what they have in mind.]