STEP 2013, Paper 1, Q12 – Solution (2 pages; 7/4/21)

(i) **1st part**

$n = k = 3$

Let the 3 types of tablet be A, B & C. Then the tablets in the container are AAABBBCCC.

P(one tablet of each type is selected on the 1st day)

$=$ P(any tablet is chosen initially)

$\times$ P(a different tablet is then chosen)

$\times$ P(a tablet different from the 1st 2 is then chosen)

$= 1 \times \frac{6}{8} \times \frac{3}{7} = \frac{9}{28}$, as required.

[Alternative method:

Prob. $= \frac{no.\ of\ ways\ of\ obtaining\ the\ letters\ ABC\ in\ some\ order}{no.\ of\ ways\ of\ placing\ 3\ letters\ out\ of\ 9\ (where\ order\ matters)}$

$= \frac{no.\ of\ ways\ of\ obtaining\ ABC\ (in\ that\ order)\times 3!}{9\times 8\times 7} = \frac{3\times 3\times 3\times 3!}{9\times 8\times 7} = \frac{9}{28}$]

2nd part

By symmetry, Prob. $= \frac{9}{28}$ also.

(ii) n days, 3 tablets

The tablets in the container are A...AB...BC...C

P(one tablet of each type is selected on the 1st day)

$=$ P(any tablet is chosen initially)

$\times$ P(a different tablet is then chosen)

$\times$ P(a tablet different from the 1st 2 is then chosen)

$= 1 \times \frac{2n}{3n-1} \times \frac{n}{3n-2} = \frac{2n^2}{(3n-1)(3n-2)}$

[Alternative method:

Prob. $= \frac{no.\ of\ ways\ of\ obtaining\ the\ letters\ ABC\ in\ some\ order}{no.\ of\ ways\ of\ placing\ 3\ letters\ out\ of\ 3n\ (where\ order\ matters)}$

$= \frac{no.\ of\ ways\ of\ obtaining\ ABC\ (in\ that\ order)\times 3!}{3n(3n-1)(3n-2)}$

$= \frac{n\times n\times n\times 3!}{3n(3n-1)(3n-2)} = \frac{2n^2}{(3n-1)(3n-2)}$]

(iii) **1st part**

n days, 2 tablets

The tablets in the container are A...AB...B

P(one tablet of each type is selected on the 1st day)

$=$ P(any tablet is chosen initially) $\times$
P(a different tablet is then chosen)

$= 1 \times \frac{n}{2n-1}$

P(one tablet of each type is selected on the 2nd day

| one tablet of each type is selected on the 1st day) $= 1 \times \frac{n-1}{2n-3}$

And so P(one tablet of each type is selected on each day)

$= \frac{n}{2n-1} \times \frac{n-1}{2n-3} \times \frac{n-2}{2n-5} \times \ldots \times \frac{1}{1}$

[Note that there are n items multiplied together on the top, and also n items multiplied together on the bottom.]

$= \frac{n!(2n)(2n-2)\ldots 2}{(2n)!} = \frac{(n!)^2 2^n}{(2n)!}$

2nd part Applying Stirling's approximation,

$\frac{(n!)^2 2^n}{(2n)!} \approx \frac{2n\pi\left(\frac{n}{e}\right)^{2n} 2^n}{\sqrt{2(2n)\pi}\left(\frac{2n}{e}\right)^{2n}} = 2^{-n}\sqrt{n\pi}$, as required.