STEP 2012, Paper 3, Q7 – Solution (3 pages; 16/7/18)

[A standard approach for dealing with simultaneous differential eq'ns is to differentiate one of the eq'ns and use the original eq'ns to eliminate the unwanted variable (in this case, z).]

$\dot{y} = -2(y - z) \Rightarrow \ddot{y} = -2(\dot{y} - \dot{z}) = -2\dot{y} + 2(-\dot{y} - 3z)$

so that $\ddot{y} = -4\dot{y} - 6z$ (1)

Also $\dot{y} = -2(y - z) \Rightarrow 2z = \dot{y} + 2y$ (2)

Then (1)&(2) $\Rightarrow \ddot{y} + 4\dot{y} + 3(\dot{y} + 2y) = 0$

and hence $\ddot{y} + 7\dot{y} + 6y = 0$

Auxiliary eq'n is $\lambda^2 + 7\lambda + 6 = 0 \Rightarrow (\lambda + 6)(\lambda + 1) = 0$

$\Rightarrow \lambda = -6 \text{ or } -1$

$\Rightarrow y = Ae^{-t} + Be^{-6t}$, as required

$\Rightarrow \dot{y} = -Ae^{-t} - 6Be^{-6t}$

Then $\dot{y} = -2(y - z) \Rightarrow \dot{y} + 2y = 2z$

$\Rightarrow z = \frac{1}{2}(-Ae^{-t} - 6Be^{-6t}) + (Ae^{-t} + Be^{-6t})$

so that $z = \frac{1}{2}Ae^{-t} - 2Be^{-6t}$, as required

(i) $z(0) = 0$ & $y(0) = 5$

$\Rightarrow \frac{A}{2} - 2B = 0 \Rightarrow A - 4B = 0$ & $A + B = 5$

$\Rightarrow 5B = 5 \Rightarrow B = 1$ & $A = 4$

So the required sol'n is: $y(t) = 4e^{-t} + e^{-6t}$

and $z_1(t) = 2e^{-t} - 2e^{-6t}$

(ii) $z(0) = 0$ & $z(1) = c$

$\Rightarrow \frac{A}{2} - 2B = c$ & $\frac{1}{2}Ae^{-1} - 2Be^{-6} = c$

$\Rightarrow \frac{1}{2}Ae^{-1} = c + e^{-6}(\frac{A}{2} - c)$

$\Rightarrow \frac{A}{2}(e^{-1} - e^{-6}) = c(1 - e^{-6})$

$\Rightarrow A = \frac{2c(1-e^{-6})}{e^{-1}-e^{-6}} = \frac{2c(e^{6}-1)}{e^{5}-1}$

& $2B = \frac{A}{2} - c = \frac{c(e^{6}-1)}{e^{5}-1} - c$

$= \frac{c\left(e^{6}-1-(e^{5}-1)\right)}{e^{5}-1} = \frac{c(e^{6}-e^{5})}{e^{5}-1}$

So the required sol'n is: $y(t) = \frac{2c(e^{6}-1)}{e^{5}-1}e^{-t} + \frac{c(e^{6}-e^{5})}{2(e^{5}-1)}e^{-6t}$

and $z_2(t) = \frac{c(e^{6}-1)}{e^{5}-1}e^{-t} - \frac{c(e^{6}-e^{5})}{(e^{5}-1)}e^{-6t}$ (1)

(iii) $\sum_{n=-\infty}^{0} z_1(t-n) = \sum_{n=-\infty}^{0}\{2e^{-(t-n)} - 2e^{-6(t-n)}\}$

$= 2e^{-t}\sum_{n=-\infty}^{0} e^{n} \ - 2e^{-6t}\sum_{n=-\infty}^{0} e^{6n}$ (2)

Let $u = -n$

Then (2) $= 2e^{-t}\sum_{u=\infty}^{0} e^{-u} \ - 2e^{-6t}\sum_{u=\infty}^{0} e^{-6u}$

$= \frac{2e^{-t}}{1-e^{-1}} - \frac{2e^{-6t}}{1-e^{-6}}$

Referring to (1), we want $\frac{c(e^{6}-1)}{e^{5}-1} = \frac{2}{1-e^{-1}}$,

so that $c = \frac{2(e^{5}-1)}{(e^{6}-1)(1-e^{-1})}$

Result to prove (again referring to (1)):

$\frac{2(e^5-1)}{(e^6-1)(1-e^{-1})} \cdot \frac{(e^6-e^5)}{(e^5-1)} = \frac{2}{1-e^{-6}}$

or that $\frac{2(e^5-1)}{(e^6-1)(1-e^{-1})} \cdot \frac{(e^6-e^5)}{(e^5-1)} - \frac{2}{1-e^{-6}} = 0$ (3)

$LHS = \frac{2}{(e^6-1)(1-e^{-1})(1-e^{-6})}\{(e^6-e^5)(1-e^{-6}) - (e^6 - 1)(1-e^{-1})\}$

Then the expression in $\{\ \} = e^6 - 1 - e^5 + e^{-1} - (e^6 - e^5 - 1 + e^{-1}) = 0$,

so that $(3) = 0$, as required

and $c = \frac{2(e^5-1)}{(e^6-1)(1-e^{-1})} = \frac{2e(e^5-1)}{(e^6-1)(e-1)}$