STEP 2012, Paper 3, Q1 – Solution (3 pages; 16/7/18)

[1st result is straightforward]

(i) [Here we have no choice but to make the most obvious use of the initial result, and see where it leads]

Let $z = y\left(\frac{dy}{dx}\right)^2$, so that $\frac{dz}{dx} = \frac{dy}{dx}\left(\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}\right)$

Then $\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = \sqrt{y}$

$\Rightarrow \frac{dz}{dx} = \frac{dy}{dx}\sqrt{y}$

$\Rightarrow z = \int \sqrt{y}\, dy = \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c \quad (*)$

[this can be verified by differentiating both sides of (*)]

Hence $\frac{2}{3}y^{\frac{3}{2}} + c = y\left(\frac{dy}{dx}\right)^2$

$x = 0, y = 1, \frac{dy}{dx} = 0 \Rightarrow \frac{2}{3} + c = 0 \Rightarrow c = -\frac{2}{3}$

So $\frac{dy}{dx} = \sqrt{\frac{2}{3}\left(y^{\frac{1}{2}} - y^{-1}\right)}$

$\Rightarrow \int 1\, dx = \int \frac{1}{\sqrt{\frac{2}{3}\left(y^{\frac{1}{2}} - y^{-1}\right)}}\, dy$

[Now see what happens if the denominator is simplified a bit]

$\Rightarrow x = \sqrt{\frac{3}{2}}\int \frac{y^{\frac{1}{2}}}{\sqrt{y^{\frac{3}{2}} - 1}}\, dy = \sqrt{\frac{3}{2}} \cdot \frac{2}{3}\int \frac{\frac{3}{2}y^{\frac{1}{2}}}{\sqrt{y^{\frac{3}{2}} - 1}}\, dy$

[the substitution $u = y^{\frac{1}{2}}$ can now be made , if necessary]

$= \sqrt{\frac{2}{3}} \cdot \frac{\sqrt{y^{\frac{3}{2}}-1}}{\left(\frac{1}{2}\right)} + c$

$x = 0, y = 1 \Rightarrow c = 0$

Hence $x^2 = \frac{8}{3}\left(y^{\frac{3}{2}} - 1\right)$,

so that $y^{\frac{3}{2}} = \frac{3}{8}x^2 + 1$, and $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$, as required.

(ii) [Once again, the simplest possible approach is the one to try – even if we can't see if it leads anywhere. Here, the only way that the expression $n\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}$ can be brought into play is if

$n = -2$]

If $z = y^{-2}\left(\frac{dy}{dx}\right)^2$, then (from (i)):

$\frac{dz}{dx} = y^{-3}\frac{dy}{dx}\left(-2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}\right)$ (A)

Then $\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} + y^2 = 0 \Rightarrow -2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2y^2$

and then $(A) \Rightarrow \frac{dz}{dx} = y^{-3}\frac{dy}{dx}(2y^2) = 2y^{-1}\frac{dy}{dx}$

and so $z = 2\int \frac{1}{y}\,dy$ [which can be verified by differentiation]

Thus $y^{-2}\left(\frac{dy}{dx}\right)^2 = 2lny + c$

$x = 0, y = 1, \frac{dy}{dx} = 0 \Rightarrow c = 0$

so that $\frac{dy}{dx} = \sqrt{2y^2lny} \Rightarrow \int 1\,dx = \int \frac{1}{\sqrt{2y^2lny}}\,dy = \frac{1}{\sqrt{2}}\int \frac{1}{y\sqrt{lny}}\,dy$

[spotting that $\int \frac{1}{y} dy = lny$:]

Let $u = lny$, so that $du = \frac{1}{y} dy$

Then $x = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{u}} du = \frac{1}{\sqrt{2}} \cdot \frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c = \sqrt{2}\,(lny)^{\frac{1}{2}} + c$

$x = 0, y = 1 \Rightarrow c = 0$

So $\frac{x^2}{2} = lny$ and hence $y = e^{\frac{x^2}{2}}$

[The 'stem' that the official Hints & Sol'ns keeps mentioning refers, I believe, to the 'stem' result that has to be shown prior to part (i). Even with this knowledge though, it doesn't make complete sense. The Examiners' Report is also not entirely clear. The printing error mentioned in the ER has presumably not been reproduced in the current version of the paper. Interestingly, the ER seems to imply that candidates should have been able to spot that something was wrong with the question! ("in spite of the printing error … two thirds of the candidates attempted this question")]