STEP 2012, Paper 3, Q13 – Solution (2 pages; 16/7/18)

(i) $E(Z|a < Z < b) = \int_a^b \frac{p_Z(u)u}{P(a<Z<b)}\, du$,

where $p_Z(u)$ is the pdf of Z

$= \frac{1}{(\Phi(b)-\Phi(a))}\int_a^b \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right) u\, du$

$= \frac{1}{\sqrt{2\pi}}\frac{1}{(\Phi(b)-\Phi(a))}\left[-\exp\left(-\frac{u^2}{2}\right)\right]_a^b$

$= \frac{\exp\left(-\frac{a^2}{2}\right)-\exp\left(-\frac{b^2}{2}\right)}{\sqrt{2\pi}(\Phi(b)-\Phi(a))}$

(ii) $Z = \frac{X-\mu}{\sigma} \Rightarrow X = \mu + \sigma Z$

So $E(X|X > 0) = E(\mu + \sigma Z|\mu + \sigma Z > 0)$

$= \mu + \sigma E(Z|Z > -\frac{\mu}{\sigma})$ (1)

$m = E(|X|) = P(X > 0)E(X|X > 0) + P(X < 0)E(-X|X < 0)$

Let $Y = -X$.

Then $E(-X|X < 0) = E(Y|Y > 0) = -\mu + \sigma E(Z|Z > \frac{\mu}{\sigma})$, from (1)

since $E(Y) = -\mu$ & $Var(Y) = \sigma^2$

So $m = P\left(Z > -\frac{\mu}{\sigma}\right)\left[\mu + \sigma E\left(Z\middle|Z > -\frac{\mu}{\sigma}\right)\right]$

$+(1 - P\left(Z > -\frac{\mu}{\sigma}\right))\left[-\mu + \sigma E\left(Z\middle|Z > \frac{\mu}{\sigma}\right)\right]$

$$= \left[1 - \Phi(-\frac{\mu}{\sigma})\right]\left[\mu + \sigma\left(\frac{exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right) - exp(-\infty)}{\sqrt{2\pi}\left(1 - \Phi(-\frac{\mu}{\sigma})\right)}\right)\right]$$

$$+\Phi(-\frac{\mu}{\sigma})\left[-\mu+\sigma\left(\frac{exp\left(-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\right)-exp(-\infty)}{\sqrt{2\pi}\left(1-\Phi(\frac{\mu}{\sigma})\right)}\right)\right]$$

[Note: $1-\Phi(-\frac{\mu}{\sigma})$ could have been written as $\Phi(\frac{\mu}{\sigma})$, but the required result involves $\Phi(-\frac{\mu}{\sigma})$]

$$=\mu\left(1-2\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\frac{\sigma}{\sqrt{2\pi}}exp\left(-\frac{1}{2}\left(-\frac{\mu}{\sigma}\right)^2\right)$$

$$+\frac{\sigma}{\sqrt{2\pi}}\left[1-\Phi(\frac{\mu}{\sigma})\right]\left(\frac{exp\left(-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\left(1-\Phi(\frac{\mu}{\sigma})\right)}\right)$$

$$=\mu\left(1-2\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\frac{2\sigma}{\sqrt{2\pi}}exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}\right)$$

$$=\mu\left(1-2\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\sigma\sqrt{\frac{2}{\pi}}exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}\right)$$, as required

$Var(|X|)=E(|X|^2)-[E(|X|)]^2=E(X^2)-m^2$

& $\sigma^2=E(X^2)-\mu^2$

so that $Var(|X|)=\sigma^2+\mu^2-m^2$