STEP 2012, Paper 3, Q11 – Solution (2 pages; 16/7/18)

(i) Taking P as the zero of PE, conservation of energy $\Rightarrow$

initial PE of rope = PE of particle (having fallen a distance x)

+ PE of rope + KE of particle + KE of rope:

$$-(2M)g\left(\frac{L}{2}\right) = -mgx$$

$$-(2M)\left(\frac{x}{2L}\right)g\left(\frac{x}{2}\right) - (2M)\left(1-\frac{x}{2L}\right)g\left(x+\frac{L+\frac{1}{2}x-x}{2}\right)$$

$$+\frac{1}{2}mv^2 + \frac{1}{2}(2M)\left(\frac{L+\frac{1}{2}x-x}{2L}\right)v^2$$

(where the PE of the rope is obtained by considering the single top section (up to x away from P) and the double bottom section separately) [as the ER mentions, it's well worth spelling this out, to keep the examiner happy]

$$\Rightarrow v^2\left\{\frac{1}{2}m + \frac{M}{4L}(2L-x)\right\} = -MgL + mgx + \frac{Mgx^2}{2L}$$

$$+\frac{1}{2L}Mg(2L-x)\left(2x+L-\frac{1}{2}x\right)$$

$$\Rightarrow v^2\left(\frac{1}{2L}\right)\left\{mL + ML - \frac{1}{2}Mx\right\} = -MgL + mgx + \frac{Mgx^2}{2L}$$

$$+\frac{Mg}{2L}(2L-x)\left(L+\frac{3x}{2}\right)$$

$$= -MgL + mgx + \frac{Mgx^2}{2L} + \frac{Mg}{2L}\left(2L^2 + 3Lx - xL - \frac{3x^2}{2}\right)$$

$$= -MgL + mgx + \frac{Mgx^2}{2L} + MgL + \frac{3Mgx}{2} - \frac{Mgx}{2} - \frac{3Mgx^2}{4L}$$

$$= mgx + Mgx - \frac{Mgx^2}{4L}$$

$$= \frac{gx}{L}\left(mL + ML - \frac{Mx}{4}\right)$$

$\Rightarrow v^2 = \frac{2gx\left(mL+ML-\frac{Mx}{4}\right)}{\left(mL+ML-\frac{1}{2}Mx\right)}$, as required (1)

Differentiating both sides of (1) wrt t:

$2v\frac{dv}{dt} = 2g\frac{d}{dt}\left\{\frac{x\left(\lambda-\frac{Mx}{4}\right)}{\left(\lambda-\frac{1}{2}Mx\right)}\right\}$, writing $\lambda = mL + ML$

so that acc. $= \frac{dv}{dt} = \frac{g}{v}\left\{\frac{\left(\lambda-\frac{1}{2}Mx\right)\left(\lambda-\frac{1}{2}Mx\right)\frac{dx}{dt}-x\left(\lambda-\frac{Mx}{4}\right)\left(-\frac{1}{2}M\right)\frac{dx}{dt}}{\left(\lambda-\frac{1}{2}Mx\right)^2}\right\}$

$= g + \frac{Mgx}{2}\frac{\left(\lambda-\frac{Mx}{4}\right)}{\left(\lambda-\frac{1}{2}Mx\right)^2}$, as $\frac{dx}{dt} = v$,

as required (once λ is replaced with $mL + ML$)

To show that acc. $> g$, we need to establish that

$mL + ML - \frac{Mx}{4} > 0$

As $x \leq 2L,\ mL + ML - \frac{Mx}{4} \geq mL + ML - \frac{ML}{2} = mL + \frac{ML}{2} > 0$