STEP 2012, Paper 2, Q12 - Solution (15/7/18; 2 pages)

(i) P(sp light on) = P(H sp switch on) × P(K sp switch on)

= (P(H switches orig. all on) × $\frac{3}{4}$ + P(H switches orig. all off) × $\frac{1}{4}$)

× (P(K sp switch orig. on)× $\frac{3}{4}$ + P(K sp switch orig. off)× $\frac{1}{4}$)

$= (p \times \frac{3}{4} + (1-p) \times \frac{1}{4})(\frac{1}{2} \times \frac{3}{4} + \frac{1}{2} \times \frac{1}{4})$

$= \frac{1}{4}(3p + 1 - p)\left(\frac{1}{2}\right)$ (*)

$= \frac{2p+1}{8}$

P(H sp switch pressed | sp light on)

$= \frac{P(\text{H sp switch pressed \& sp light on})}{P(\text{sp light on})}$

P(H sp switch pressed & sp light on)

= P(H sp switch orig. off)×P(H sp switch pressed)

×P(correct K sw pressed)

$= (1-p) \times \frac{1}{4} \times \frac{1}{2}$ (the factor of ½ is the same as the ½ in (*))

So P(H sp switch pressed | sp light on) $= \frac{(1-p)\times\frac{1}{8}}{\frac{2p+1}{8}} = \frac{1-p}{1+2p}$, as required.

(ii) [This is a very strange and unclear situation. If in doubt, assume the simplest possible interpretation: in this case, that we are dealing with a random variable $\sim B(7,\frac{1-p}{1+2p})$; even though it sounds as though we are considering 7 Mondays in the past.]

Let X be the number of times that the H sp switch was pressed (given that the sp light was on).

Then $X \sim B(7,\frac{1-p}{1+2p})$

It is given that $P(X=2) < P(X=3)$

and $P(X=4) < P(X=3)$, as 3 is the modal value of X

Noting that $1-\left(\frac{1-p}{1+2p}\right) = \frac{1+2p-1+p}{1+2p} = \frac{3p}{1+2p}$,

$^7C_2\left(\frac{1-p}{1+2p}\right)^2\left(\frac{3p}{1+2p}\right)^5 < {}^7C_3\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4$

and $^7C_4\left(\frac{1-p}{1+2p}\right)^4\left(\frac{3p}{1+2p}\right)^3 < {}^7C_3\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4$

so that $\frac{7(6)}{2}\left(\frac{3p}{1+2p}\right) < \frac{7(6)(5)}{6}\left(\frac{1-p}{1+2p}\right)$

and $\frac{1-p}{1+2p} < \frac{3p}{1+2p}$

Hence $3(3p) < 5(1-p)$ and $1-p < 3p$,

so that $14p < 5$ and $1 < 4p$,

giving $p < \frac{5}{14}$ and $p > \frac{1}{4}$

ie $\frac{1}{4} < p < \frac{5}{14}$, as required