STEP 2012, Paper 2, Q11 - Solutions (15/7/18; 2 pages)

Conservation of momentum $\Rightarrow \sum_{i=1}^{n} im.\frac{u}{i} = (km + \sum_{i=1}^{n} im)v$

$\Rightarrow v = \frac{nu}{k+\sum_{i=1}^{n} i} = \frac{nu}{k+\frac{1}{2}n(n+1)} = \frac{2nu}{2k+n(n+1)}$, as required

Let X be the number of collisions.

Then $\frac{u}{X+1} \leq \frac{2Xu}{N(N+1)+X(X+1)}$ (1)

Considering equality: $N(N+1) + X(X+1) = 2X(X+1)$,

so that $N(N+1) = X(X+1)$

Thus $X = N$ satisfies (1); ie N collisions occur.

Initial KE of particles $= \frac{1}{2}\sum_{i=1}^{N}(im)\left(\frac{u}{i}\right)^2$

Final KE of particles & block $= \frac{1}{2}(km + \sum_{i=1}^{N} im)\left(\frac{2Nu}{2k+N(N+1)}\right)^2$

Therefore loss of KE

$= \frac{1}{2}\, mu^2 \sum_{i=1}^{N} \frac{1}{i}$

$-\frac{1}{2}m\left(\frac{1}{2}N(N+1) + \frac{1}{2}N(N+1)\right).\frac{4N^2u^2}{(N(N+1)+N(N+1))^2}$

$= \left(\frac{1}{2}mu^2 \sum_{i=1}^{N} \frac{1}{i}\right) - \frac{1}{2}mu^2.\frac{4N^2}{4N(N+1)}$

$= \frac{1}{2}mu^2 \left\{\left(\sum_{i=1}^{N} \frac{1}{i}\right) - \frac{N}{N+1}\right\}$

$= \frac{1}{2}mu^2 \left\{\left(\sum_{i=2}^{N+1} \frac{1}{i}\right) - \frac{1}{N+1} + 1 - \frac{N}{N+1}\right\}$

$= \frac{1}{2} m u^2 \left(\sum_{n=2}^{N+1} \frac{1}{n} \right)$, as required