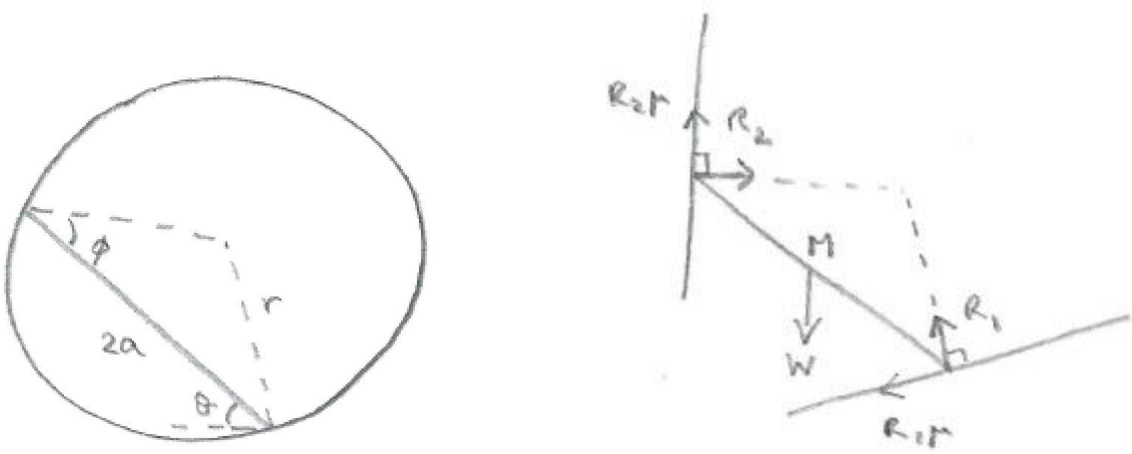
STEP 2012, Paper 2, Q10 - Solutions (15/7/18; 2 pages)



[3 equations can be created, by taking moments and resolving in 2 perpendicular directions. As we don't want the weight of the rod to appear in the equations, it seems natural to take moments about the midpoint of the rod, and then to resolve horizontally. This gives the 1st result asked for in the question. But unfortunately the 2nd result cannot be readily obtained in this way. Instead, it turns out that, by resolving along & perp. to the rod, the weight can be eliminated, and the required result obtained.]

[Note: It is also possible, in general, to take moments about 2 points and resolve in 1 direction, or to take moments about 3 points; but information can sometimes be lost (eg, in the latter case, if the 3 points lie on a straight line).]

Taking moments about M,

$$aR_1 sin\phi - aR_1\mu cos\phi - aR_2\mu cos\phi - aR_2 sin\phi = 0$$

$$\Rightarrow \mu cos\phi(R_1 + R_2) = sin\phi(R_1 - R_2)$$

$\Rightarrow \mu(R_1 + R_2) = (R_1 - R_2)tan\phi$, as required (1)

Resolving along the rod:

$R_1 cos\phi + R_1 \mu sin\phi - Wsin\theta - R_2 cos\phi + R_2 \mu sin\phi = 0$ (2)

Resolving perp. to the rod:

$R_1 sin\phi - R_1 \mu cos\phi - Wcos\theta + R_2 sin\phi + R_2 \mu cos\phi = 0$ (3)

(2)&(3) $\Rightarrow \frac{Wsin\theta}{Wcos\theta} = \frac{R_1 cos\phi + R_1 \mu sin\phi - R_2 cos\phi + R_2 \mu sin\phi}{R_1 sin\phi - R_1 \mu cos\phi + R_2 sin\phi + R_2 \mu cos\phi}$

$= \frac{R_1 - R_2 + \mu(R_1 + R_2)tan\phi}{(R_1 + R_2)tan\phi - (R_1 - R_2)\mu}$

$= \frac{(R_1 - R_2)tan\phi + \mu(R_1 + R_2)tan^2\phi}{(R_1 + R_2)tan^2\phi - (R_1 - R_2)\mu tan\phi}$ [in order to be able to use (1)]

$= \frac{\mu(R_1 + R_2) + \mu(R_1 + R_2)tan^2\phi}{(R_1 + R_2)tan^2\phi - \mu^2(R_1 + R_2)}$, from (1)

$= \frac{\mu(1 + tan^2\phi)}{tan^2\phi - \mu^2}$

Then $cos\phi = \frac{a}{r}$, so that $tan^2\phi = sec^2\phi - 1 = \frac{r^2}{a^2} - 1$

So $tan\theta = \frac{\mu\left(\frac{r^2}{a^2}\right)}{\frac{r^2}{a^2} - 1 - \mu^2} = \frac{\mu r^2}{r^2 - a^2(1 + \mu^2)}$, as required

[The wording of the last part of the question (ie "Deduce …") is arguably misleading, as the 2nd result isn't used.]

From (1), $tan\lambda = \frac{(R_1 - R_2)tan\phi}{(R_1 + R_2)} < tan\phi$

and hence $\lambda < \phi$ (as $0 < \phi < \frac{\pi}{2}$)

[Given the two misleading aspects mentioned, this is a fairly treacherous question!]