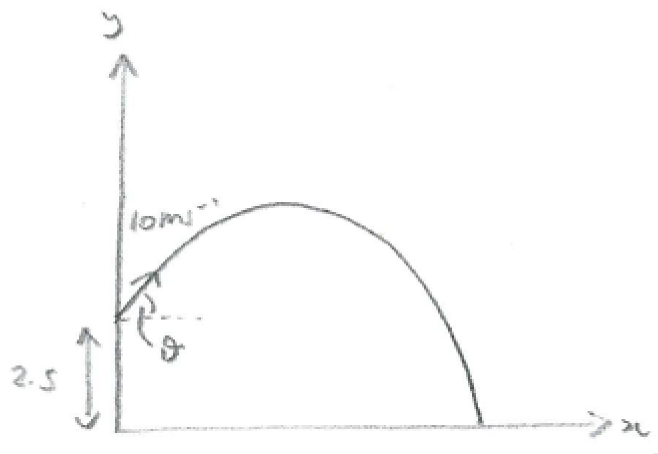
STEP 2012, Paper 1, Q9 – Solution (3 pages; 15/6/18)

Please note: The official 2012 "Hints and Answers" are mainly hints rather than answers, so the following sol'n hasn't been checked against anything.



$y = 2.5 + ut + \frac{1}{2}at^2 \Rightarrow$

when shot hits the ground, $0 = 2.5 + 10sin\theta.t - 5t^2$

$\Rightarrow 10t^2 - 20sin\theta.t - 5 = 0$

$\Rightarrow 2t^2 - 4sin\theta.t - 1 = 0$

$\Rightarrow t = \frac{4sin\theta \pm \sqrt{16sin^2\theta + 8}}{4} = sin\theta + \frac{1}{2}\sqrt{4sin^2\theta + 2}$, rejecting $t < 0$

Then, as $sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$,

$t = \frac{\sqrt{1-c}}{\sqrt{2}} + \frac{1}{2}\sqrt{2(1-c) + 2} = \frac{1}{\sqrt{2}}(\sqrt{1-c} + \sqrt{2-c})$,

as required

Range $= 10cos\theta\ .t = \frac{10cos\theta}{\sqrt{2}}(\sqrt{1-c} + \sqrt{2-c})$

Let $R' = cos\theta(\sqrt{1-c} + \sqrt{2-c})$

The range is maximised when $\frac{dR'}{d\theta} = 0$

$\frac{dR'}{d\theta} = -sin\theta\left(\sqrt{1-c} + \sqrt{2-c}\right)$

$+\cos\theta\{\frac{1}{2}(1-c)^{-\frac{1}{2}}(-1) \ + \frac{1}{2}(2-c)^{-\frac{1}{2}}(-1)\}\frac{dc}{d\theta}$

$= -sin\theta\left(\sqrt{1-c} + \sqrt{2-c}\right)$

$+\cos\theta\left\{\frac{1}{2}(1-c)^{-\frac{1}{2}}(-1) \ + \frac{1}{2}(2-c)^{-\frac{1}{2}}(-1)\right\}(-1)\sin(2\theta)(2)$

$\frac{dR'}{d\theta} = 0 \Rightarrow -sin\theta\left(\sqrt{1-c} + \sqrt{2-c}\right)$

$+\cos\theta\sin(2\theta)\left\{\frac{1}{\sqrt{1-c}} + \frac{1}{\sqrt{2-c}}\right\} = 0$

$\Rightarrow \frac{sin\theta}{\sqrt{1-c}\sqrt{2-c}}\{-(1-c)\sqrt{2-c} - (2-c)\sqrt{1-c}$

$+2cos^2\theta[\sqrt{2-c} + \sqrt{1-c}\,]\} = 0$

$\Rightarrow \sqrt{2-c}\,(c-1+1+c) + \sqrt{1-c}\,(c-2+1+c) = 0$,

as $2cos^2\theta = 1+c$ (with $sin\theta = 0$ giving the minimum range)

$\Rightarrow 2c\sqrt{2-c} \ + \sqrt{1-c}\,(2c-1) = 0$

$\Rightarrow 2c\sqrt{2-c} = \sqrt{1-c}\,(1-2c)$

$\Rightarrow 4c^2(2-c) = (1-4c+4c^2)(1-c)$

$\Rightarrow 8c^2 = 1-4c+4c^2-c+4c^2$

$\Rightarrow 0 = 1-5c$

$\Rightarrow c = 1/5$, as required

Let $R(c) = \frac{10cos\theta}{\sqrt{2}}(\sqrt{1-c} + \sqrt{2-c})$

$= \frac{10\sqrt{\frac{1+c}{2}}}{\sqrt{2}}\left(\sqrt{1-c} + \sqrt{2-c}\right) = 5\sqrt{1+c}\left(\sqrt{1-c} + \sqrt{2-c}\right)$

The required extra distance $= R\left(\frac{1}{5}\right) - R(0)$ (as $\cos\big(2(45)\big) = 0$)

$= 5\sqrt{\frac{6}{5}}\left(\sqrt{\frac{4}{5}} + \sqrt{\frac{9}{5}}\right) - 5(1 + \sqrt{2}\,)$

$= 2\sqrt{6} + 3\sqrt{6} - 5(1 + \sqrt{2})$

$= 5(\sqrt{6} - \sqrt{2} - 1)$ m , as required