STEP 2012, Paper 1, Q5 – Solution (2 pages; 15/6/18)

Please note: The official 2012 "Hints and Answers" are mainly hints rather than answers, so the following sol'n hasn't been checked against anything.

Let $u = cosx$, so that $du = -sinx\ dx$

and $\int_0^{\frac{\pi}{4}} sin(2x)ln(cosx)dx = -2\int_1^{\frac{1}{\sqrt{2}}} ulnu\ du$

Then, by Parts (integrating u):

$= -2\left[\frac{1}{2}u^2lnu\right]_1^{\frac{1}{\sqrt{2}}} + 2\int_1^{\frac{1}{\sqrt{2}}}\frac{1}{2}u^2\left(\frac{1}{u}\right)\ du$

$= -\frac{1}{2}ln\left(\frac{1}{\sqrt{2}}\right) + \int_1^{\frac{1}{\sqrt{2}}} u\ du$

$= \frac{1}{2}ln\sqrt{2} + \left[\frac{1}{2}u^2\right]_1^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{4}ln2 + \frac{1}{4} - \frac{1}{2}$

$= \frac{1}{4}(ln2 - 1)$, as required

[Unfortunately, the above substitution doesn't help for the next part.] By Parts (integrating $\cos(2x)$),

$\int_0^{\frac{\pi}{4}} cos(2x)ln(cosx)dx$

$= \left[\frac{1}{2}\sin(2x)\ln(cosx)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\frac{1}{2}\sin(2x)\frac{1}{cosx}(-sinx)dx$

$= \frac{1}{2}ln\left(\frac{1}{\sqrt{2}}\right) + \int_0^{\frac{\pi}{4}} sin^2x\ dx$

$= -\frac{1}{2}ln\sqrt{2} + \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos(2x)\ dx$

$= -\frac{1}{4}ln2 + \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{8}ln4 + \frac{1}{2}(\frac{\pi}{4} - \frac{1}{2})$

$= \frac{1}{8}(\pi - ln4 - 2)$, as required

Last part:

$cosx + sinx = \sqrt{2}\cos(x - \frac{\pi}{4})$

Let $\theta = x - \frac{\pi}{4}$

so that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\{cos(2x) + \sin(2x)\}ln(cosx + sinx)dx$

$= \int_0^{\frac{\pi}{4}}\left\{\cos\left(2\theta + \frac{\pi}{2}\right) + sin\left(2\theta + \frac{\pi}{2}\right)\right\}\ln\{\sqrt{2}cos\theta\}\ d\theta$

$= \int_0^{\frac{\pi}{4}}\{-\sin(2\theta) + \cos(2\theta)\}\ln\{cos\theta\}\ d\theta$

$+ \int_0^{\frac{\pi}{4}}\{-\sin(2\theta) + \cos(2\theta)\}\ln\sqrt{2}\ d\theta$

$= \frac{1}{8}(\pi - ln4 - 2) - \frac{1}{4}(ln2 - 1)$

$+ \frac{1}{2}ln2[\frac{1}{2}\cos(2\theta) + \frac{1}{2}\sin(2\theta)]_0^{\frac{\pi}{4}}$ (using the earlier results)

$= \frac{\pi}{8} - \frac{1}{4}ln2 - \frac{1}{4} - \frac{1}{4}ln2 + \frac{1}{4} + \frac{1}{4}ln2\{[0 + 1] - [1 + 0]\}$

$= \frac{\pi}{8} - \frac{1}{2}ln2$