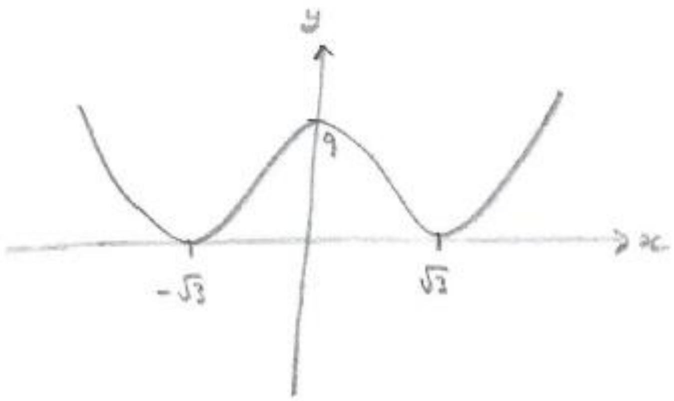
STEP 2012, Paper 1, Q2 – Solution (4 pages; 15/6/18)

Please note: The official 2012 "Hints and Answers" are mainly hints rather than answers, so the following sol'n hasn't been checked against anything.

(i) $y = x^4 - 6x^2 + 9 = (x^2 - 3)^2$ (symmetry about y-axis)

& $\frac{dy}{dx} = 4x(x^2 - 3) \Rightarrow$ stationary points at (0,9), $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$



[When the Hints & Sol'ns say "by considering the graph of $y = x^2 - 3$", they are referring to the fact that the squaring of $x^2 - 3$ causes the portion below the x- axis to be reflected in the x- axis (as well as squaring the magnitude).]

Considering vertical translations of $y = x^4 - 6x^2 + 9$ [it is perhaps easier to visualise the x-axis moving up and down, than the curve moving]:

(a) $(n = 0)$ $b > 9$

(b) $(n = 1)$ none

(c) $(n = 2)$ $b = 9$ or $b < 0$

(d) $(n = 3)$ $b = 0$

(e) $(n = 4)$ $0 < b < 9$

[Note that when the Hints & Solutions refer to "the second part", they mean this last part; not part (ii)]

(ii) $y = x^4 - 6x^2 + ax + b$

$\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 12x + a = 0$ (1)

$\frac{d^2y}{dx^2} = 0 \Rightarrow 12x^2 - 12 = 0 \Rightarrow x = \pm 1$

From (1), $x = 1 \Rightarrow a = 8$ and $x = -1 \Rightarrow a = -8$

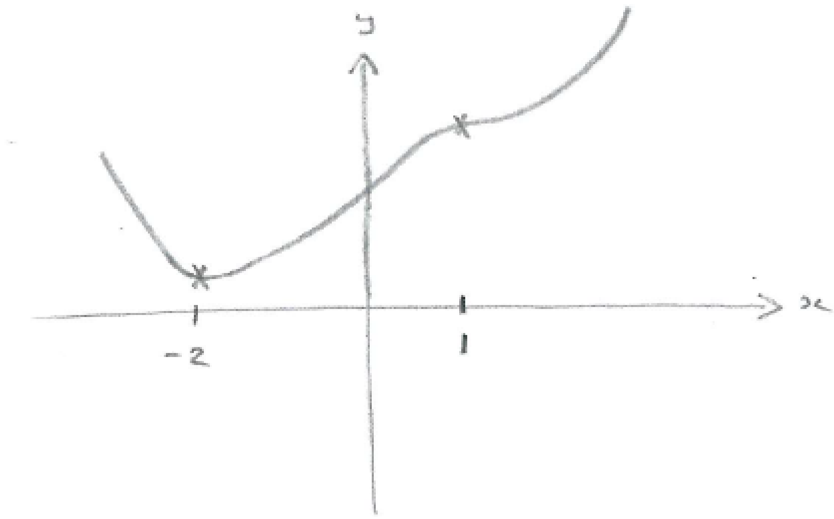
When $a = 8$, $\frac{dy}{dx} = 4x^3 - 12x + 8$

We know that $x = 1$ is a root of $\frac{dy}{dx} = 0$,

and so $4x^3 - 12x + 8 = 4(x^3 - 3x + 2) = 4(x-1)(x^2 + x - 2)$

$= 4(x-1)^2(x+2)$

[$\frac{d^2y}{dx^2} = 0 \Rightarrow$ point of inflexion if $\frac{d^3y}{dx^3} \neq 0$ (this is a sufficient, but not necessary condition); in this case, $\frac{d^3y}{dx^3} = 24x$, so $\frac{d^3y}{dx^3} \neq 0$ at $x = \pm 1$]



Now $y(-2) = 16 - 24 - 16 + b = b - 24$,

so that $b < 24 \Rightarrow$ no roots of $x^4 - 6x^2 + ax + b = 0$

$b = 24 \Rightarrow 1$ root

$b > 24 \Rightarrow 2$ roots

[again, it is probably easier to visualise the x-axis moving up and down]

When $a = -8$, $\frac{dy}{dx} = 4x^3 - 12x - 8$

$x = -1$ is a root of $\frac{dy}{dx} = 0$,

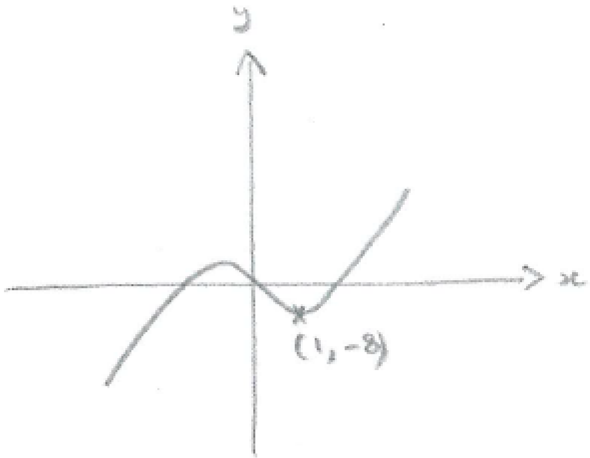
so that $4x^3 - 12x - 8 = 4(x^3 - 3x - 2) = 4(x + 1)(x^2 - x - 2)$

$= 4(x + 1)^2(x - 2)$

and $y(2) = 16 - 24 - 16 + b = b - 24$, as before; so the same cases arise

(iii) $y = x^4 - 6x^2 + ax \Rightarrow \frac{dy}{dx} = 4x^3 - 12x + a$ (1)

Consider the graph of $y = 4x^3 - 12x = 4x(x^2 - 3)$ (2)

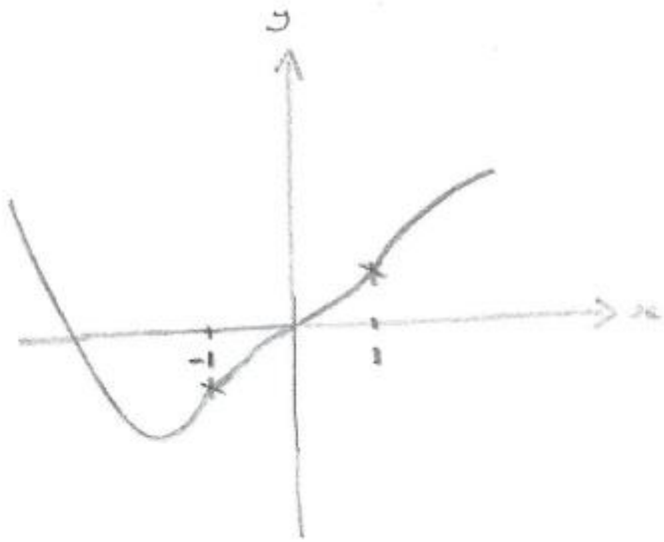


Then $\frac{d}{dx}(4x^3 - 12x) = 12x^2 - 12$, so that the turning points of (2) are at $x = \pm 1$

As the minimum is at $(1, -8)$, (1) will cross the x-axis once when $a > 8$; ie $y = x^4 - 6x^2 + ax$ will have one stationary point (for a value of $x < -1$).

Also $\frac{d}{dx}(4x^3 - 12x + a) = 12x^2 - 12 = 0$ at $x = \pm 1$ (as already established), so that there will be points of inflexion of

$y = x^4 - 6x^2 + ax$ at these values.



[To help draw non-stationary points of inflexion, think of $tanx$ at $x = 0$, where $\frac{dy}{dx} = 1$]