STEP 2012, Paper 1, Q11 – Solution (3 pages; 15/6/18)

Please note: The official 2012 "Hints and Answers" are mainly hints rather than answers, so the following sol'n hasn't been checked against anything.

(i) If T is the tension in the string, by N2L:

A: $T = 5mgsin\theta + \mu(5mg)cos\theta$ (1)

B: $T = 3mgsin\phi + \mu(3mg)cos\phi$ (2)

P: $2T = Mg$ (3)

where $tan\theta = \frac{7}{24}$, so that $sin\theta = \frac{7}{\sqrt{24^2-7^2}} = \frac{7}{25}$ & $cos\theta = \frac{24}{25}$

and $tan\phi = \frac{4}{3}$, so that $sin\phi = \frac{4}{5}$ & $cos\phi = \frac{3}{5}$

Then (1) & (2) ⇒ $\frac{7}{5} + \frac{24\mu}{5} = \frac{12}{5} + \frac{9\mu}{5}$,

so that $15\mu = 5$ & $hence\ \mu = \frac{1}{3}$

Also, (1) & (3) ⇒ $Mg = mg(\frac{14}{5} + \frac{16}{5}) \Rightarrow M = 6m$, as required.

(ii) Let $x_A(t)$, $x_B(t)$ & $x_P(t)$ be the distances of A, B & P from the fixed pulleys at time t.

Then $x_P(t) = x_P(0) + \frac{1}{2}\left(x_A(0) - x_A(t) + x_B(0) - x_B(t)\right)$, since for P to drop by a given distance, the lengths QP and RP must both increase by that distance, so that the total extra length of string needed is twice the distance fallen by P.

[Note that, depending on the relative masses of A and B and the angles of the slopes, the accelerations of A and B will differ, and their contributions to the extra length of string will also differ. In the H&A, it mentions that the acceleration is not constant, but presumably this refers to the above discussion, rather than any variation over time.]

So $\ddot{x}_P(t) = -\frac{1}{2}(\ddot{x}_A(t) + \ddot{x}_B(t))$

Thus, if the accelerations of A and B up the slope are a_A & a_B, and the acceleration downwards of P is a_P, then

$a_P = \frac{1}{2}(a_A + a_B)$

The tension throughout the string will still be the same (you wouldn't be expected to justify this), so that N2L now gives:

A: $T - 5mgsin\theta - \mu(5mg)cos\theta = 5ma_A$ (4)

B: $T - 3mgsin\phi - \mu(3mg)cos\phi = 3ma_B$ (5)

P: $9mg - 2T = 9ma_P$ (6)

These give:

$2T = 9mg - 9ma_P = 10ma_A + 10mgsin\theta + \mu(10mg)cos\theta$

& $9mg - 9ma_P = 6ma_B + 6mgsin\phi + \mu(6mg)cos\phi$

$\Rightarrow 9g - \frac{9}{2}(a_A + a_B) = 10a_A + 10g\left(\frac{7}{25}\right) + \left(\frac{1}{3}\right)(10g)\left(\frac{24}{25}\right)$

& $9g - \frac{9}{2}(a_A + a_B) = 6a_B + 6g\left(\frac{4}{5}\right) + \left(\frac{1}{3}\right)(6g)\left(\frac{3}{5}\right)$

$\Rightarrow 90g - 45(a_A + a_B) = 100a_A + 28g + 32g$

& $90g - 45(a_A + a_B) = 60a_B + 48g + 12g$

$\Rightarrow 30g = 145a_A + 45a_B$

& $30g = 45a_A + 105a_B$

$\Rightarrow 6g = 29a_A + 9a_B$

& $6g = 9a_A + 21a_B$

$\Rightarrow a_A = \frac{\begin{vmatrix} 6g & 9 \\ 6g & 21 \end{vmatrix}}{\begin{vmatrix} 29 & 9 \\ 9 & 21 \end{vmatrix}} = \frac{6g(12)}{3(203-27)} = \frac{24g}{176} = \frac{3g}{22}$

& $a_B = \frac{\begin{vmatrix} 29 & 6g \\ 9 & 6g \end{vmatrix}}{\begin{vmatrix} 29 & 9 \\ 9 & 21 \end{vmatrix}} = \frac{6g(20)}{3(203-27)} = \frac{40g}{176} = \frac{5g}{22}$

so that $a_P = \frac{1}{2}(a_A + a_B) = \frac{4g}{22}$