STEP 2011, Paper 3, Q7 – Solution (3 pages; 12/6/18)

(i) When $n = 2$,

$T_n = (\sqrt{a+1} + \sqrt{a})^2 = (a+1) + a + 2\sqrt{(a+1)a}$

$= A_2 + B_2\sqrt{(a+1)a}$,

where $A_2 = 2a + 1$ & $B_2 = 2$

And $a(a+1){B_2}^2 + 1 = 4a^2 + 4a + 1 = (2a+1)^2 = {A_2}^2$,

so that the required result is true for $n = 2$

Assume now that the result is true for $n = k$ (even).

We wish to show that the result will then be true for $n = k + 2$

$T_{k+2} = (A_k + B_k\sqrt{a(a+1)})(A_2 + B_2\sqrt{a(a+1)})$

$= A_kA_2 + B_kB_2a(a+1) + (A_kB_2 + B_kA_2)\sqrt{a(a+1)}$

Setting $B_{k+2} = A_kB_2 + B_kA_2$ and ${A_{k+2}}^2 = a(a+1){B_{k+2}}^2 + 1$,

we need to show that $A_{k+2} = A_kA_2 + B_kB_2a(a+1)$,

or equivalently that

$\{A_kA_2 + B_kB_2a(a+1)\}^2 = a(a+1){B_{k+2}}^2 + 1$

or that ${A_k}^2{A_2}^2 + {B_k}^2{B_2}^2a^2(a+1)^2 + 2A_kA_2B_kB_2a(a+1)$

$-a(a+1)(A_kB_2 + B_kA_2)^2 - 1 = 0$ (1)

$LHS\ of\ (1)$

$= {A_k}^2{A_2}^2 + {B_k}^2{B_2}^2a^2(a+1)^2 + 2A_kA_2B_kB_2a(a+1)$

$-a(a+1){A_k}^2{B_2}^2 - a(a+1){B_k}^2{A_2}^2 - 2a(a+1)A_kB_2B_kA_2 - 1$

$= \{a(a+1){B_k}^2+1\}\{a(a+1){B_2}^2+1\}$

$+{B_k}^2{B_2}^2a^2(a+1)^2 - a(a+1)\{a(a+1){B_k}^2+1\}{B_2}^2$

$-a(a+1){B_k}^2\{a(a+1){B_2}^2+1\}-1$

$= (1-1)+a(a+1){B_k}^2\{1-1\}+a(a+1){B_2}^2\{1-1\}$

$+a^2(a+1)^2{B_k}^2{B_2}^2\{1+1-1-1\}=0,$ as required.

Thus, if the given result is true for $n=k$, it is true for $n=k+2$

As it is true for $n=2$, it follows that it is true for $n=4,6,8,\ldots$, and hence by the principle of induction it is true for all even n.

(ii) If $n=1$, then $C_n=D_n=1$, and $(a+1){C_n}^2=a+1$,

whilst $a{D_n}^2+1=a+1$ also.

If n is odd and >1, then $T_n=(\sqrt{a+1}+\sqrt{a})(\sqrt{a+1}+\sqrt{a})^{n-1}$, which from (i) can be written in the form

$(\sqrt{a+1}+\sqrt{a})(A_{n-1}+B_{n-1}\sqrt{a(a+1)})$,

(where ${A_{n-1}}^2=a(a+1){B_{n-1}}^2+1$) (2)

$=\sqrt{a+1}\{A_{n-1}+aB_{n-1}\}+\sqrt{a}\{A_{n-1}+(a+1)B_{n-1}\}$

Setting $C_n=A_{n-1}+aB_{n-1}$

and $D_n=A_{n-1}+(a+1)B_{n-1}$,

We want to show that $(a+1){C_n}^2=a{D_n}^2+1$ (3)

$(a+1){C_n}^2-a{D_n}^2-1$

$=(a+1)(A_{n-1}+aB_{n-1})^2-a(A_{n-1}+(a+1)B_{n-1})^2-1$

$= {A_{n-1}}^2(a+1-a) + {B_{n-1}}^2\{(a+1)a^2 - a(a+1)^2\}$

$+2A_{n-1}B_{n-1}\{(a+1)a - a(a+1)\} - 1$

$= {A_{n-1}}^2 + {B_{n-1}}^2(a+1)a(a-[a+1]) - 1$

$= {A_{n-1}}^2 - {B_{n-1}}^2(a+1)a - 1$

$= 0, \text{ from } (2)$

Thus (3) is satisfied.

(iii) When n is even, ${A_n}^2 - {B_n}^2 a(a+1) = 1$

and when n is odd, ${C_n}^2(a+1) - {D_n}^2 a = 1$,

as required.