STEP 2011, Paper 3, Q13 – Solution (2 pages; 12/6/18)

$P(X = r) = \frac{k!}{r!(k-r)!}\left(\frac{b}{n}\right)^r\left(1 - \frac{b}{n}\right)^{k-r}$

Let $A(r) = \frac{P(X=r+1)}{P(X=r)} = \frac{\frac{k!}{(r+1)!(k-r-1)!}\left(\frac{b}{n}\right)^{r+1}\left(1-\frac{b}{n}\right)^{k-r-1}}{\frac{k!}{r!(k-r)!}\left(\frac{b}{n}\right)^{r}\left(1-\frac{b}{n}\right)^{k-r}}$

$= \frac{r!(k-r)!\left(\frac{b}{n}\right)}{(r+1)!(k-r-1)!(1-\frac{b}{n})} = \frac{(k-r)b}{(r+1)(n-b)}$

We want R such that $A(R-1) > 1$ & $A(R) < 1$,

so that $\frac{(k-R+1)b}{R(n-b)} > 1$ & $\frac{(k-R)b}{(R+1)(n-b)} < 1$

$\Rightarrow kb - Rb + b > Rn - Rb$ & $kb - Rb < Rn - Rb + n - b$

$\Rightarrow R < \frac{b(k+1)}{n}$ & $R > \frac{kb+b-n}{n} = \frac{b(k+1)}{n} - 1$

So, if there is a unique solution, $\frac{b(k+1)}{n} \notin \mathbb{Z}$

and $R = \left\lfloor\frac{b(k+1)}{n}\right\rfloor$

The answer isn't unique if $A(R) = 1$ for some R

$\Rightarrow kb - Rb = Rn - Rb + n - b$

$\Rightarrow R = \frac{kb+b-n}{n} = \frac{b(k+1)}{n} - 1$

which will be the case if $\frac{b(k+1)}{n} \in \mathbb{Z}$; ie if n divides $b(k+1)$

(ii) $P(X = r) = \frac{N_1}{N_2}$

where N_1 = number of ways of choosing r black balls from b and $k - r$ non-black balls from $n - b$, and $N_2 =$ number of ways of choosing k balls from n

$$= \frac{\binom{b}{r}\binom{n-b}{k-r}}{\binom{n}{k}}$$

Let $B(r) = \frac{P(X=r+1)}{P(X=r)} = \frac{\left(\frac{b!(n-b)!k!(n-k)!}{(r+1)!(b-r-1)!(k-r-1)!(n-b-k+r+1)!n!}\right)}{\left(\frac{b!(n-b)!k!(n-k)!}{r!(b-r)!(k-r)!(n-b-k+r)!n!}\right)}$

$$= \frac{r!(b-r)!(k-r)!(n-b-k+r)!n!}{(r+1)!(b-r-1)!(k-r-1)!(n-b-k+r+1)!n!}$$

$$= \frac{(b-r)(k-r)}{(r+1)(n-b-k+r+1)}$$

Then, as in (i), we want R such that $B(R-1) > 1$ & $B(R) < 1$,

so that $(b-R+1)(k-R+1) > R(n-b-k+R)$

& $(b-R)(k-R) < (R+1)(n-b-k+R+1)$

$\Rightarrow bk - bR + b - Rk + R^2 - R + k - R + 1 > Rn - Rb - Rk + R^2$

& $\quad bk - bR - Rk + R^2$

$< Rn - Rb - Rk + R^2 + R + n - b - k + R + 1$

$\Rightarrow bk + b - 2R + k + 1 > Rn$

& $\quad bk < Rn + 2R + n - b - k + 1$

$\Rightarrow R < \frac{bk+b+k+1}{n+2}$ & $R > \frac{bk-n+b+k-1}{n+2}$

ie $R < \frac{(b+1)(k+1)}{n+2}$ & $R > \frac{(b+1)(k+1)}{n+2} - 1$

and by the same reasoning as in (i), $R = \left\lfloor \frac{(b+1)(k+1)}{n+2} \right\rfloor$

and the answer isn't unique if $B(R) = 1$ for some R; ie when

$n + 2$ divides $(b+1)(k+1)$