STEP 2011, Paper 3, Q12 – Solution (3 pages; 12/6/18)

[The standard results for the pgf of X, $G(t) = \sum_{k=0}^{\infty} p_k t^k$ are that $E(X) = G'(1)$ & $Var(X) = G''(1) + G'(1) - [G'(1)]^2$ (as can be seen by differentiating $\sum_{k=0}^{\infty} p_k t^k$ twice).]

$E(Y) = \frac{d}{dt} G\big(H(t)\big)\big| t = 1 = G'\big(H(t)\big)H'(t)\big| t = 1$

[Note that $G'\big(H(t)\big)$ means $\frac{d}{d\lambda} G(\lambda)$, where $\lambda = H(t)$]

Then, as $H(1) = 1$, $E(Y) = G'(1)H'(1) = E(N)E(X_i)$

And $Var(Y) = \frac{d^2}{dt^2} G\big(H(t)\big)|(t = 1) + E(Y) - [E(Y)]^2$

where $\frac{d^2}{dt^2} G\big(H(t)\big) = \frac{d}{dt}\{G'\big(H(t)\big)H'(t)\}$

$= \big\{G''\big(H(t)\big)H'(t)\big\}H'(t) + G'\big(H(t)\big)H''(t)$

so that

$\frac{d^2}{dt^2} G\big(H(t)\big)|(t = 1) = \big\{G''\big(H(1)\big)H'(1)\big\}H'(1) + G'\big(H(1)\big)H''(1)$

$= G''(1)[E(X_i)]^2 + G'(1)\{Var(X_i) - E(X_i) + [E(X_i)]^2\}$

$= \{Var(N) - E(N) + [E(N)]^2\}[E(X_i)]^2$

$+E(N)\{Var(X_i) - E(X_i) + [E(X_i)]^2\}$

and $Var(Y) = \{Var(N) - E(N) + [E(N)]^2\}[E(X_i)]^2$

$+E(N)\{Var(X_i) - E(X_i) + [E(X_i)]^2\}$

$+E(N)E(X_i) - [E(N)E(X_i)]^2$

$= Var(N)[E(X_i)]^2 + E(N)Var(X_i)$

[This is, of course, a fairly standard result.]

For the coin, N has a Geometric distribution, and therefore (from the Formulae booklet), a pgf of $G(t) = \frac{0.5t}{1-0.5t} = \frac{t}{2-t}$

$Y = \sum_{i=1}^{N} X_i$, where $X_i \sim B(1,0.5)$

Thus the X_i have pgf $H(t) = 1 - 0.5 + 0.5t = \frac{1}{2}(1+t)$

(again from the Formulae booklet).

Then, applying the earlier results, the pgf of Y is

$$G\big(H(t)\big) = \frac{\frac{1}{2}(1+t)}{2-\frac{1}{2}(1+t)} = \frac{1+t}{4-(1+t)} = \frac{1+t}{3-t}$$

$E(N) = \frac{1}{\left(\frac{1}{2}\right)} = 2$ (from the Formulae booklet) [quicker than finding $G'(1)$]

$E(Y) = E(N)E(X_i) = 2(\frac{1}{2}) = 1$

$Var(N) = \frac{1-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$ (again from the Formulae booklet)

$Var(X_i) = (1)(0.5)(0.5) = 1/4$

So $Var(Y) = Var(N)[E(X_i)]^2 + E(N)Var(X_i)$

$= 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) = 1$

And $P(Y = r) =$ coefficient of t^r in

$$\frac{1+t}{3-t} = \frac{1}{3}(1+t)\left(1-\frac{t}{3}\right)^{-1} = \frac{1}{3}(1+t)(1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\cdots)$$

For $r = 0$, coefficient of t^r is $\frac{1}{3}$

For $r > 0$, coefficient of t^r is

$$\frac{1}{3}\left\{\left(\frac{1}{3}\right)^r + \left(\frac{1}{3}\right)^{r-1}\right\} = \left(\frac{1}{3}\right)^r\left\{\frac{1}{3}+1\right\} = \frac{4}{3}\left(\frac{1}{3}\right)^r = \frac{4}{3^{r+1}}$$