STEP 2011, Paper 3, Q10 – Solution (3 pages; 12/6/18)

Let x_P & x_Q be the displacements of P & Q (see Fig. 1).

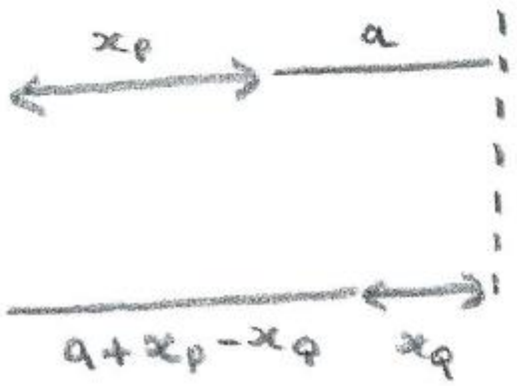

Fig. 1

We want to find x_P when $a + x_P - x_Q = a$; ie when $x_P = x_Q$

Let $v_P = \dot{x}_P$ & $v_Q = \dot{x}_Q$ be the velocities of the particles.

By conservation of energy,

$\frac{1}{2}mu^2 = \frac{1}{2}m{v_P}^2 + \frac{1}{2}m{v_Q}^2 + \frac{1}{2}k(x_P - x_Q)^2$, (1)

where k is the stiffness of the string (the last term being the elastic PE)

Also, by conservation of momentum,

$mu = mv_P + mv_Q$ (2)

And $m\ddot{x}_P = -k(x_P - x_Q)$, provided that $x_P - x_Q > 0$ (3)

Integrating (2), $ut = x_P + x_Q + A$

and $t = 0, x_P = 0, x_Q = 0 \Rightarrow A = 0$

so that $ut = x_P + x_Q$ (4)

Then, combining (1) & (3), and writing $\ddot{x}_P = \dot{v}_P$,

$\frac{1}{2}mu^2 = \frac{1}{2}m{v_P}^2 + \frac{1}{2}m{v_Q}^2 + \frac{1}{2}k(\frac{m\dot{v}_P}{k})^2$

and so from (2), $u^2 = {v_P}^2 + (u - v_P)^2 + \frac{m}{k}(\dot{v}_P)^2$,

giving $0 = 2{v_P}^2 - 2uv_p + \frac{m}{k}(\dot{v}_P)^2$,

and hence $\frac{dv_p}{dt} = \sqrt{\frac{2k}{m}(uv_p - {v_P}^2)}$

$\Rightarrow t\sqrt{\frac{2k}{m}} = \int \frac{1}{\sqrt{uv_p - {v_P}^2}} dt$ (5)

Now, $uv_p - {v_P}^2 = -\left(v_p - \frac{u}{2}\right)^2 + \frac{u^2}{4}$

Also $k = \frac{\frac{1}{2}ma\omega^2}{a}$, so that $\sqrt{\frac{2k}{m}} = \omega$,

and so (5) $\Rightarrow \omega t = arcsin\left(\frac{v_p - \frac{u}{2}}{\frac{u}{2}}\right) + B$

and hence $\sin(\omega t - B) = \frac{v_p - \frac{u}{2}}{\frac{u}{2}}$,

giving $v_p = \frac{u}{2}\sin(\omega t - B) + \frac{u}{2}$

Then $t = 0, v_p = u \Rightarrow u = \frac{u}{2}\sin(-B) + \frac{u}{2}$

$\Rightarrow -B = \frac{\pi}{2}$ and so $v_p = \frac{u}{2}\sin\left(\omega t + \frac{\pi}{2}\right) + \frac{u}{2}$

or $v_p = \frac{u}{2}(\cos(\omega t) + 1)$ (6)

Integrating (6): $x_p = \frac{u\omega}{2}\sin(\omega t) + \frac{u}{2}t + C$

and $t = 0, x_p = 0 \Rightarrow C = 0$

and so $x_p = \frac{u\omega}{2}\sin(\omega t) + \frac{u}{2}t$ (7)

As mentioned earlier, when the string is next of length a,

$x_P = x_Q, t = T_1(say) \Rightarrow$ (from (4)) $uT_1 = 2x_P$ (8)

Thus, from (7), $\frac{u\omega}{2}\sin(\omega T_1) = 0$, and so $T_1 = \frac{\pi}{\omega}$

and from (8), $x_P = \frac{uT_1}{2} = \frac{u\pi}{2\omega}$, as required.

From (6), $v_p = \frac{u}{2}(\cos(\omega t) + 1)$,

and so at $t = T_1, v_p = 0$

Then, from (2), $v_Q = u$

After time T_1, the string is slack, and so the particles collide after a further time $\frac{a}{u}$; ie at total time $\frac{\pi}{\omega} + \frac{a}{u}$