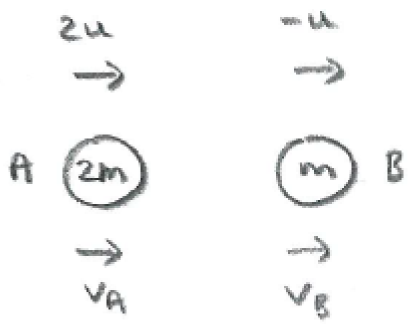
STEP 2011, Paper 2, Q9 – Solution (4 pages; 12/6/18)

[The first two parts of this question are surprisingly straightforward - especially for Paper 2. But the algebra at the end makes up for this to some extent.]



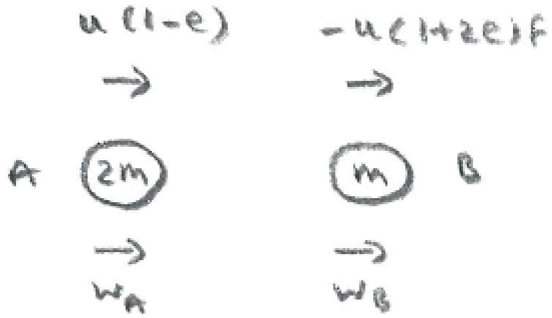
Conservation of momentum ⇒ $2m(2u) + m(-u) = 2mv_A + mv_B$

$\Rightarrow 3u = 2v_A + v_B \quad (1)$

Newton's law of impact ⇒ $v_B - v_A = e\big(2u - (-u)\big) = 3eu \quad (2)$

Then $(1) - (2) \Rightarrow 3v_A = 3u - 3eu \Rightarrow v_A = u(1-e)$

and then $(1) \Rightarrow v_B = 3u - 2u(1-e) = u(1+2e)$



Conservation of momentum ⇒

$(2m)u(1-e) + m(-u)(1+2e)f = (2m)w_A + mw_B$

$\Rightarrow 2u(1-e) - u(1+2e)f = 2w_A + w_B \quad (3)$

Newton's law of impact ⇒

$w_B - w_A = e[u(1-e) + u(1+2e)f]$ (4)

(3) $\Rightarrow w_A = u(1-e) - \frac{u}{2}(1+2e)f - \frac{w_B}{2}$

(4) $\Rightarrow w_A = w_B - eu[(1-e) + (1+2e)f]$

so that

$u(1-e) - \frac{u}{2}(1+2e)f - \frac{w_B}{2} = w_B - eu[(1-e) + (1+2e)f]$

and hence

$\frac{3w_B}{2} = u(1-e) - \frac{u}{2}(1+2e)f + eu[(1-e) + (1+2e)f]$

$\Rightarrow w_B = \frac{2}{3}u\{(1-e) + e(1-e)\} + \frac{2}{3}uf\{-\frac{1}{2}(1+2e) + e(1+2e)\}$

$= \frac{2}{3}u(1-e^2) + \frac{1}{3}uf\{-(1+2e) + 2e(1+2e)\}$

$= \frac{2}{3}(1-e^2)u + \frac{1}{3}uf(4e^2-1)$

giving the required expression.

We need to show that $w_B > 0$,

or equivalently that $h(e,f) = 2(1-e^2) - (1-4e^2)f > 0$

(i) One option is to try any obvious rearrangement

Thus $h(e,f) = e^2(-2+4f) + 2 - f$

$= 4e^2f + (1-f) + (1-2e^2)$

The first two terms are non-negative, but the third would be negative if eg $e = 1$.

(ii) Although outside the syllabus, we could consider finding the minimum of $h(e,f)$ using partial differentiation. However, there is no guarantee that such a minimum exists: it could be the case that $h(e,f)$ attains its lowest value at a boundary point of the domain of e & f (ie where either e or f is 0 or 1).

(iii) We could explore extreme situations. Clearly the result to be proved depends on one or more of the following:

$e > 0, f > 0, 1 - e \geq 0, 1 - f \geq 0$

Thus we see that $h(0,0) = 2, h(0,1) = 1, h(1,0) = 0$

& $h(1,1) = 3$

A critical point is thus $e = 1, f = 0$

To simplify the working, we could write $g = 1 - e^2$, and expect to use the facts that $f > 0$ & $g \geq 0$

Then $2(1 - e^2) - (1 - 4e^2)f = 2g - f + 4(1 - g)f$

$= 2g + 3f - 4fg$

As $1 - g > 0$ & $1 - f \geq 0$,

we can write

$2g + 3f - 4fg = Af(1 - g) + Bg(1 - f) + Cf + Dg$ (*)

where $A + C = 3, B + D = 2, -A - B = -4$, with

A, B, C & $D \geq 0$, provided that at least one of A & C is > 0 (as

$f(1 - g) > 0$ & $f > 0$, but $g(1 - f) \geq 0$ & $g \geq 0$, and we want (*) to be > 0)

For example, $B = 2, D = 0, A = 2$ & $C = 1$

so that (*) becomes $2f(1 - g) + 2g(1 - f) + f$, which is > 0 from the discussion above

(Alternatively, $C = 0, A = 3, B = 1$ & $D = 1$)

Alternatively, we can see that if $1 - 4e^2 < 0$, so that $\frac{1}{4} < e^2$ and $\frac{1}{2} < e$, then $2(1 - e^2) \geq 0$ and $-(1 - 4e^2)f > 0$, so that $h(e, f) > 0$

If $e = \frac{1}{2}$, then $2(1 - e^2) > 0$ and $-(1 - 4e^2)f = 0$, which again gives $h(e, f) > 0$

If $e < \frac{1}{2}$, let $g = 1 - 4e^2$, so that $0 < g < 1$

Then $h(e, f) = 2 + \frac{1}{2}(g - 1) - gf = \frac{1}{2}(3 + g - 2gf)$

$= \frac{1}{2}(g + 2(1 - gf) + 1)$

which is > 0, as $gf < 1$