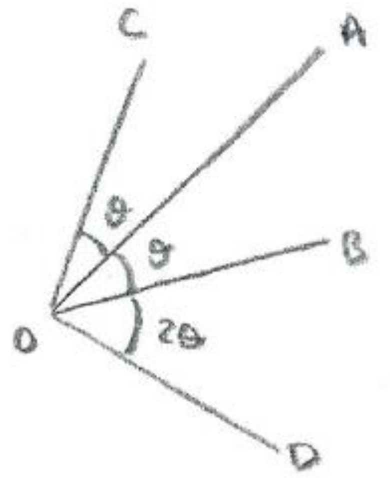
STEP 2011, Paper 2, Q5 – Solution (2 pages; 12/6/18)



(The diagram is for an acute θ only.)

$\underline{c} = k\underline{a} - (\underline{b} - k\underline{a})$, for some real k

$= 2k\underline{a} - \underline{b}$

$= \lambda\underline{a} - \underline{b}$ (say)

Also, $\underline{a}.\underline{b} = cos\theta = \underline{a}.\underline{c} = \underline{a}.\left(\lambda\underline{a} - \underline{b}\right) = \lambda\underline{a}.\underline{a} - \underline{a}.\underline{b}$,

so that $2\underline{a}.\underline{b} = \lambda\underline{a}.\underline{a}$ and $\lambda = \frac{2\underline{a}.\underline{b}}{\underline{a}.\underline{a}}$

2nd part

From the 1st part, $\underline{d} = \theta\underline{b} - \underline{c}$, where $\theta = \frac{2\underline{b}.\underline{c}}{\underline{b}.\underline{b}}$

$= \theta\underline{b} - \left(\lambda\underline{a} - \underline{b}\right) = (\theta + 1)\underline{b} - \lambda\underline{a}$

$\Rightarrow \mu = \frac{2\underline{b}.\underline{c}}{\underline{b}.\underline{b}} + 1 = \frac{2\underline{b}.(\lambda\underline{a} - \underline{b})}{\underline{b}.\underline{b}} + 1$

$= \frac{2\lambda\underline{b}.\underline{a}}{\underline{b}.\underline{b}} - 1 = \frac{4(\underline{a}.\underline{b})^2}{(\underline{a}.\underline{a})(\underline{b}.\underline{b})} - 1$

3rd part

In general, points on the line through $\underline{a}$ & $\underline{b}$ can be expressed as $\gamma\underline{a} + (1-\gamma)\,\underline{b}$; thus, as $\underline{d}$ is such a point, $\mu - \lambda = 1$

4th part

$cosAOB = cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$

From the 2nd part, $cos^2\theta = \frac{\mu+1}{4}$

If $\lambda = -\frac{1}{2}, \mu = -\frac{1}{2} + 1 = \frac{1}{2}$

So $cos^2\theta = \frac{3}{8}$

As $\lambda < 0, cos\theta < 0,$ and so , $cos\theta = -\sqrt{\frac{3}{8}} = -\frac{\sqrt{6}}{4}$

5th part

As $\underline{d} = \frac{1}{2}\underline{b} + \frac{1}{2}\underline{a}$, D lies midway between A and B.