STEP 2011, Paper 2, Q1 – Solution (5 pages; 22/5/18)

(i) For $y = f(x) = \sqrt{1-x} + \sqrt{3+x}$, the domain is limited to $-3 \le x \le 1$

To investigate symmetry, suppose that $f(a-\lambda) = f(a+\lambda)$, for a fixed a and any λ.

Then $\sqrt{1-a+\lambda} + \sqrt{3+a-\lambda} = \sqrt{1-a-\lambda} + \sqrt{3+a+\lambda}$

If this is to work for any λ, then we want $1-a = 3+a$, so that $a = -1$, and thus there is symmetry about $x = -1$.

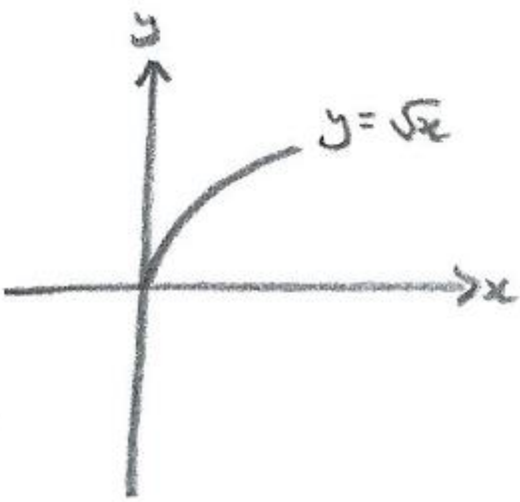
(With hindsight, $x = -1$ is a likely candidate for symmetry; being the midway point of the domain.)

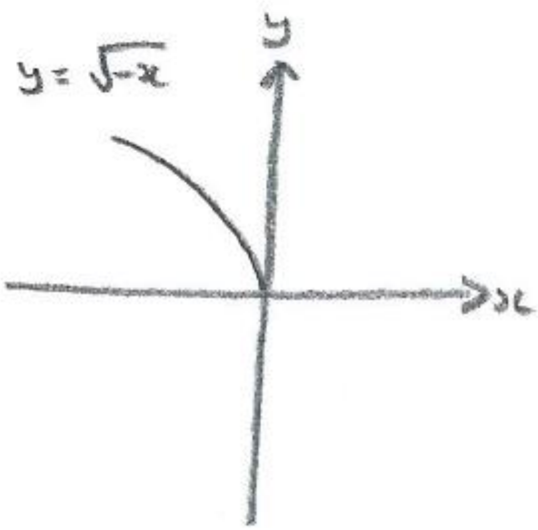
$y = f(x)$ can be obtained from $y = \sqrt{x}$ as follows:

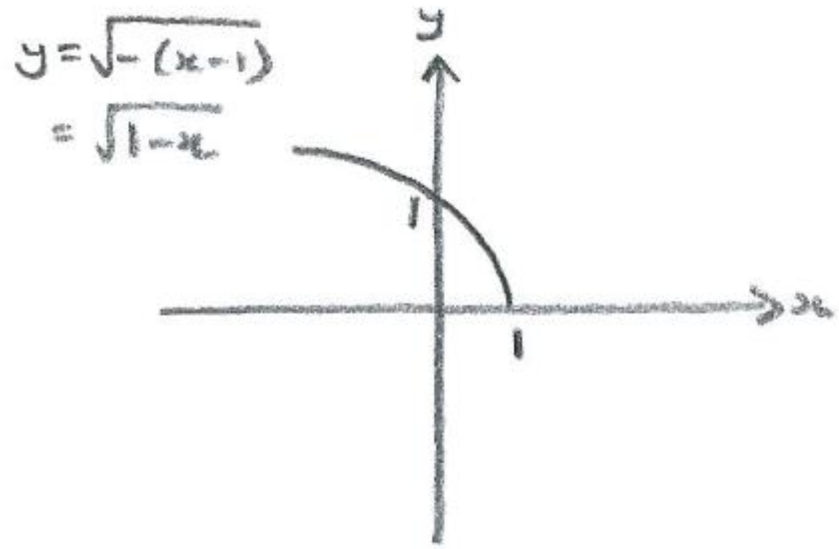
Reflect $y = \sqrt{x}$ in the y-axis to give $y = \sqrt{-x}$; then translate by $\binom{1}{0}$ to give $y = \sqrt{-(x-1)} = \sqrt{1-x}$

Also, $y = \sqrt{3+x}$ is obtained from $y = \sqrt{x}$ by a translation of $\binom{-3}{0}$

[see diagrams below]



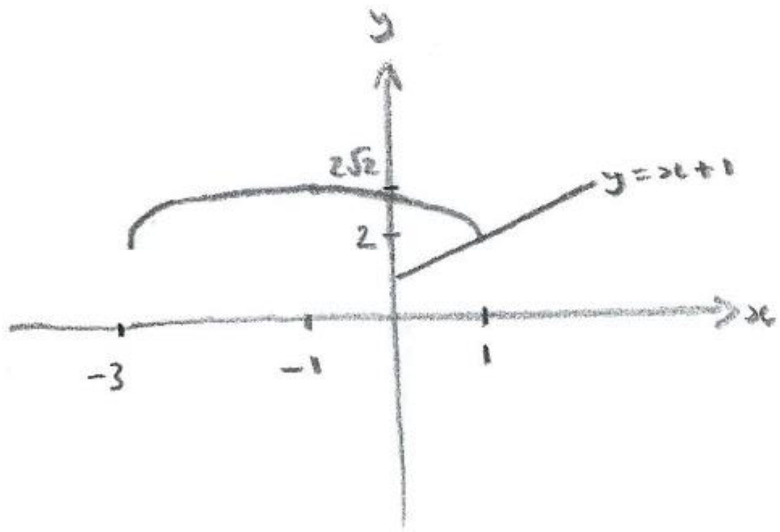




Then $y = \sqrt{1-x} + \sqrt{3+x}$ can be obtained (roughly) by adding the two graphs; noting the y values at the line of symmetry, on the y-axis, and at the end points of the domain.

Thus, $f(-1) = 2\sqrt{2}, f(0) = f(-2) = 1 + \sqrt{3}\ (< 2\sqrt{2})$

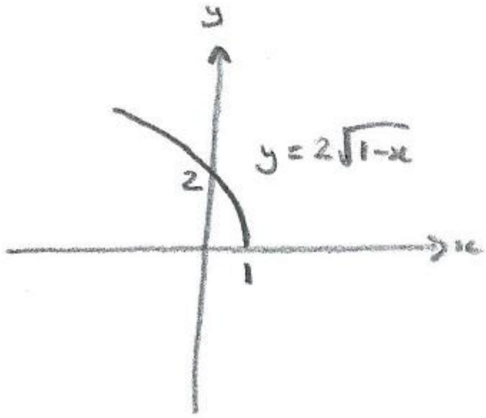
& $f(-3) = f(1) = 2\ (< 1 + \sqrt{3})$

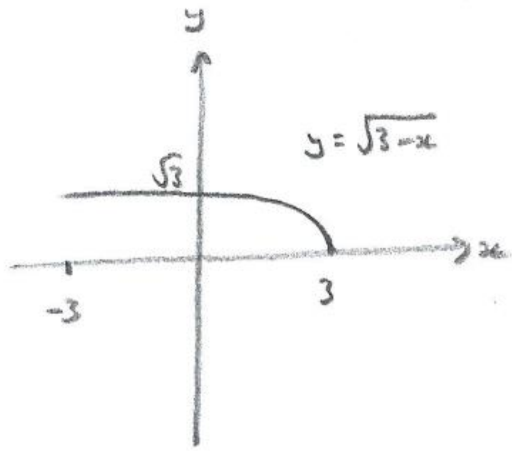


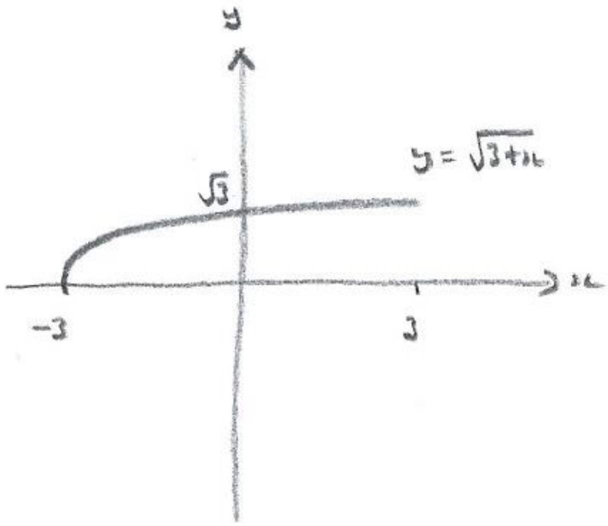
We can see that the line $y = x + 1$ cuts the curve only once, and from the end point found, this is when $x = 1$. [If in doubt, make the simplest assumption; ie that the intersection can be found easily by experimentation.]

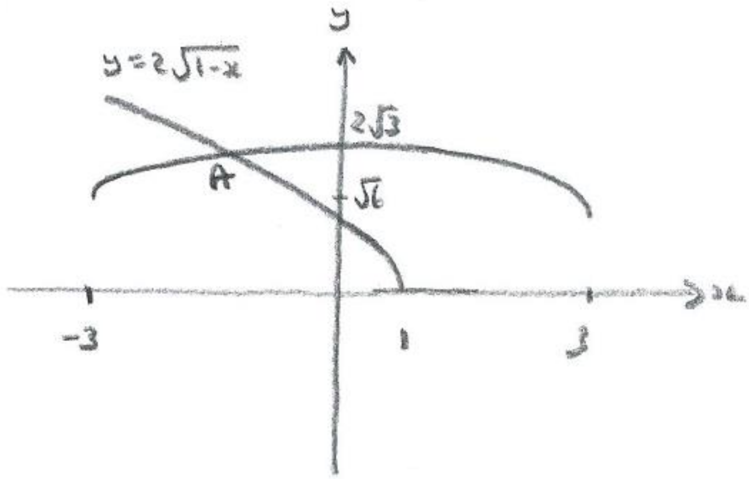
[As indicated in the official solutions, finding the gradient at the end points is a secondary issue, which the examiners are not particularly interested in. The infinite gradient could be deduced from the gradients of the components: $y = \sqrt{1-x}$ & $y = \sqrt{3+x}$]

(ii) [Here we have the usual dilemma of whether we should be using the result from the previous part, or extending the method. Having briefly considered the possibility of using the previous result, the default assumption really has to be one of treating the second part on its own merits.]









Here the symmetry of $y = \sqrt{3+x} + \sqrt{3-x}$ is at $x = 0$, and we see that there is just one point of intersection, at A.

Solving the equation algebraically:

$2\sqrt{1-x} = \sqrt{3+x} + \sqrt{3-x}$

$\Rightarrow 4(1-x) = 3 + x + 3 - x + 2\sqrt{9-x^2}$

$\Rightarrow -2 - 4x = 2\sqrt{9-x^2}$

$\Rightarrow (1+2x)^2 = 9 - x^2$

$\Rightarrow 5x^2 + 4x - 8 = 0$

$\Rightarrow x = \frac{-4 \pm \sqrt{16+160}}{10} = \frac{-4 \pm 4\sqrt{1+10}}{10}$

[The squaring process is liable to introduce spurious solutions; as $A = B \Rightarrow A^2 = B^2$, but $A = -B\ also\ \Rightarrow A^2 = B^2$]

As the point A has a negative x-value, we can reject the positive root, to give $x = -\frac{2}{5}(1 + \sqrt{11})$

[As a check, we note that $|x| < \frac{2}{5}(1 + 4) = 2 < 3$, so that we are within the domain of $y = \sqrt{3 + x} + \sqrt{3 - x}$]