STEP 2011, Paper 2, Q12 – Solution (2 pages; 12/6/18)

(i) The possibilities are:

XYY + stop

XYX + start again

YY + stop

YXY + start again

So $w = p(1-p)p + p(1-p)(1-p)w + (1-p)p$

$+(1-p)(1-p)(1-p)w$

$\Rightarrow w\{1 - p(1-p)^2 - (1-p)^3\} = p(1-p)\{p+1\}$

$\Rightarrow w = \frac{p(1-p^2)}{p^3(-1+1)+p^2(2-3)+p(-1+3)+1-1}$

(as $p \neq 0$; when $p = 0$, the denominator $= 0$)

$= \frac{p(1-p^2)}{-p^2+2p}$

$= \frac{1-p^2}{2-p}$

2nd part:

$w - \frac{1}{2} = \frac{1-p^2}{2-p} - \frac{1}{2} = \frac{2-2p^2-2+p}{2(2-p)} = \frac{p(1-2p)}{2(2-p)}$

So if $p < \frac{1}{2}, w > \frac{1}{2}$ & $if\ p > \frac{1}{2}, w < \frac{1}{2}$

3rd part:

Consider $\frac{dw}{dp} = \frac{1}{(2-p)^2}\{(2-p)(-2p) - (1-p^2)(-1)\}$

$= \frac{1}{(2-p)^2}\{p^2(2-1) - 4p + 1\}$

$= \frac{1}{(2-p)^2}\{p^2 - 4p + 1\}$

The graph of $y = p^2 - 4p + 1$ crosses the p-axis at

$\frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$, and so $\frac{dw}{dp} > 0$ when $0 < p < 2 - \sqrt{3}$

Thus w increases when p decreases and is less than $2 - \sqrt{3}$

(ii) When $p = \frac{2}{3}$, $w = \frac{1 - \frac{4}{9}}{2 - \frac{2}{3}} = \frac{(\frac{5}{9})}{(\frac{4}{3})} = \frac{5}{12}$

For the game to be fair, the expected value of Xavier's [and therefore Younis's] winnings must be zero.

So $(1 - w)(1) + w(-k) = 0$

and hence $k = \frac{1-w}{w} = \frac{\left(\frac{7}{12}\right)}{\left(\frac{5}{12}\right)} = \frac{7}{5} = 1.4$

(iii) If $p = 0$, the only possible scenario is YXY + start again; ie the match continues indefinitely

[The Examiner's report says that this question was only attempted by around a quarter of the candidates, and that the average score was very low. You may well conclude that Probability questions are a good area to specialise in, in order to gain an advantage over other candidates.]