STEP 2011, Paper 1, Q10 - Solution (3 pages; 8/4/21)

1st part

[The question only makes any sense if B is dropped after A has been dropped.]

As the bounce is perfectly elastic, no energy is lost by A.

Let M & m be the masses of A and B, and u_A & u_B their speeds at the point of collision.

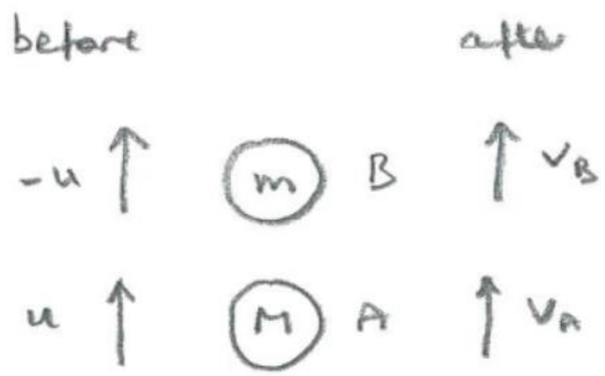
Then, by conservation of energy (assuming no air resistance!),

$Mgd = \frac{1}{2}Mu_A{}^2$ and $mgd = \frac{1}{2}mu_B{}^2$, where d is the distance from P to the point of collision.

Hence, $gd = \frac{1}{2}u_A{}^2$ and $gd = \frac{1}{2}u_B{}^2$, so that $u_A{}^2 = u_B{}^2$;

ie the two speeds are the same.

2nd part



By conservation of momentum, $Mu + m(-u) = Mv_A + mv_B$

and by Newton's law of restitution, $v_B - v_A = u - (-u)$

[We want to show that $v_A > 0$. It is self-evident that $v_B > v_A$, by the nature of the collision, but the official sol'ns seem to think that we need to prove that $v_B > 0$.]

Eliminating v_B: $(M-m)u = Mv_A + m(2u + v_A)$

$\Rightarrow v_A = \frac{(M-3m)u}{M+m} > 0$, as $M > 3m$

3rd part

Method 1

The additional energy given to B as a result of the collision is

$\frac{1}{2}m({v_B}^2 - u^2)$

[It doesn't matter that the direction of motion of B has been reversed. Had B been moving downwards with speed v_B after the collision, then it would be moving upwards with speed v_B by the time it returned to the same point, after bouncing on the plane - assuming conservation of energy.]

If h_1 is the extra height gained by B, as a result of the extra energy, then $mgh_1 = \frac{1}{2}m({v_B}^2 - u^2)$, so that $h_1 = \frac{{v_B}^2 - u^2}{2g}$

From the 2nd part, $v_B = 2u + v_A = 2u + \frac{(M-3m)u}{M+m}$,

so that ${v_B}^2 = \frac{u^2}{(M+m)^2}(2(M+m) + (M-3m))^2$

$= \frac{u^2}{(M+m)^2}(3M-m)^2$

Then $h_1 = \frac{u^2}{2g(M+m)^2}((3M-m)^2 - (M+m)^2)$

$= \frac{u^2}{2g(M+m)^2}(8M^2 - 8Mm)$

$= \frac{4u^2M(M-m)}{g(M+m)^2}$

and so the maximum height attained by B (before the 2nd collision) is

$h + \frac{4u^2M(M-m)}{g(M+m)^2}$, as required.

Method 2

Let Q be the point where the particles collide, and let R be the highest point reached by B.

Then, from $'v^2 = u^2 + 2as'$,

$0 = {v_B}^2 + 2(-g)QR$

From the 2nd part, $v_B = 2u + v_A = 2u + \frac{(M-3m)u}{M+m}$,

so that ${v_B}^2 = \frac{u^2}{(M+m)^2}(2(M+m) + (M-3m))^2$

$= \frac{u^2}{(M+m)^2}(3M-m)^2$

And the required maximum height is $H = h - PQ + QR$,

where $PQ = d$ from the 1st part, with $gd = \frac{1}{2}{u_B}^2 = \frac{1}{2}u^2$

So $H = h - \frac{u^2}{2g} + \frac{{v_B}^2}{2g} = h + \frac{u^2}{2g}\{\frac{1}{(M+m)^2}(3M-m)^2 - 1\}$

$= h + \frac{u^2}{2g(M+m)^2}((3M-m)^2 - (M+m)^2)$

$= h + \frac{u^2}{2g(M+m)^2}(8M^2 - 8Mm)$

$= h + \frac{4u^2M(M-m)}{g(M+m)^2}$

[The official sol'ns claim that we need to show that the 2nd collision occurs after B has attained its maximum height.
But $v_B > v_A$ (by the nature of the collision), so that this fact is self-evident.]