STEP 2010, Paper 3, Q9 – Solution (2 pages; 10/6/18)

As P is moving in a circular path,

$mgsin\theta - R = \frac{mv^2}{a}$ (1) ,

where R is the normal reaction of the block on P and v is the tangential speed of P at angle θ

As the string is inextensible, Q has speed v at the same moment.

Applying conservation of energy at angle θ:

gain in kinetic energy = decrease in potential energy

As the particles started at rest,

gain in kinetic energy $= \frac{1}{2}(m + M)v^2$

At angle θ, P (and hence Q) has moved a distance of $a\theta$

So decrease in potential energy = $Mga\theta - mgasin\theta$

and hence $\frac{1}{2}(m + M)v^2 = Mga\theta - mgasin\theta$ (2)

Then equating expressions for v^2 from (1) & (2) gives:

$\frac{a(mgsin\theta - R)}{m} = \frac{Mga\theta - mgasin\theta}{\frac{1}{2}(m+M)}$

$\Rightarrow mgsin\theta - R = \frac{2mg(M\theta - msin\theta)}{m+M}$

$\Rightarrow R = \frac{mg}{m+M}\{(m + M)sin\theta - 2M\theta + 2msin\theta\}$

$= \frac{mg}{m+M}\{(3m + M)sin\theta - 2M\theta\}$ (3)

As P remains in contact with the block, $R \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

$From\ (3), R \geq 0 \Rightarrow (3m + M)sin\theta - 2M\theta \geq 0$

$\Rightarrow 3 + \frac{M}{m} \geq \frac{2M\theta}{msin\theta}$ for $\theta > 0$, since $sin\theta > 0$

(and $R = 0$ for $\theta = 0$, from (3))

Thus $\frac{M}{m}(\frac{2\theta}{sin\theta} - 1) \leq 3$

and hence $\frac{m}{M} \geq \frac{1}{3}\left(\frac{2\theta}{sin\theta} - 1\right)$ (4)

For small θ, $\frac{\theta}{sin\theta} \approx 1$, so that $\frac{1}{3}\left(\frac{2\theta}{sin\theta} - 1\right) \approx \frac{1}{3}$

For $\theta = \frac{\pi}{2}$, $\frac{1}{3}(\pi - 1)$

Also $\frac{\theta}{sin\theta}$, and hence $\frac{1}{3}\left(\frac{2\theta}{sin\theta} - 1\right)$, increases with θ

So the greatest value of $\frac{1}{3}\left(\frac{2\theta}{sin\theta} - 1\right)$ for $0 < \theta \leq \frac{\pi}{2}$

is $\frac{1}{3}(\pi - 1)$

and hence, as (4) holds for $0 < \theta \leq \frac{\pi}{2}$, $\frac{m}{M} \geq \frac{1}{3}(\pi - 1)$, as required.