STEP 2010, Paper 3, Q12 – Solution (2 pages; 9/4/21)

1st part

$S = 1 + (1 + d)r + (1 + 2d)r^2 + \cdots + (1 + nd)r^n + \cdots$

Then $S - rS = 1 + r(1 + d - 1) + r^2(1 + 2d - 1 - d)$

$+ \cdots + r^n(1 + nd - 1 - (n - 1)d) + \cdots$

$= 1 + dr + dr^2 + \cdots + dr^n + \cdots$

so that $S(1 - r) = 1 + \frac{dr}{1-r}$, and hence $S = \frac{1}{1-r} + \frac{rd}{(1-r)^2}$, as required.

2nd part

[This is just the expected value of a Geometric variable.]

Expected number of shots it takes for Arthur to hit the target

$\sum_{k=1}^{\infty} k\,(1 - a)^{k-1}a = a + 2(1 - a)a + 3(1 - a)^2 a + \cdots$

$+ k(1 - a)^{k-1}a + \cdots$

$= a\{1 + 2(1 - a) + \cdots + k(1 - a)^{k-1} + \cdots\}$

$= aS$, where $r = 1 - a$ & $d = 1$.

so that the expected number of shots is

$a\left\{\frac{1}{a} + \frac{1-a}{a^2}\right\} = a.\frac{1}{a^2} = \frac{1}{a}$, as required.

[Alternatively: $\sum_{k=1}^{\infty} k\,(1 - a)^{k-1}a = -a\frac{d}{da}\sum_{k=1}^{\infty}(1 - a)^k$

$= -a\frac{d}{da}\left\{\frac{1-a}{1-(1-a)}\right\} = -a\frac{d}{da}\left\{\frac{1-a}{a}\right\} = -a(-1)a^{-2} = \frac{1}{a}$]

3rd part

$$\alpha = \sum_{r=1}^{\infty}\{P(\text{neither person has won before Arthur's rth attempt})$$

$\times$ P(Arthur then hits the target on the rth attempt)}

$= \sum_{r=1}^{\infty} (a'b')^{r-1} a = \frac{a}{1-a'b'}$, as required

(the sum to infinity of a Geometric series with common ratio $a'b'$)

[The official sol'n uses an elegant recurrence argument:

$\alpha =$ P(A wins on 1st attempt) + P(wins after 1st attempt)

$= a + a'b'\alpha$, so that $\alpha(1 - a'b') = a$, and hence $\alpha = \frac{a}{1-a'b'}$]

4th part

As Arthur or Boadicea will eventually hit the target, one of them must win the contest, and so $\alpha + \beta = 1$,

and hence $\beta = 1 - \frac{a}{1-a'b'} = \frac{1-a'b'-a}{1-a'b'} = \frac{a'-a'b'}{1-a'b'} = \frac{a'(1-b')}{1-a'b'}$

$= \frac{a'b}{1-a'b'}$

[Alternatively,

$\beta = \sum_{r=1}^{\infty} \{P(\text{neither person has won before Boadicea's rth attempt})$

$\times$ P(Boadicea then hits the target on the rth attempt)}

$= \sum_{r=1}^{\infty} [(a'b')^{r-1} a'] b$

$= \frac{a'b}{1-a'b'}$]

5th part

Expected no. of shots

$=$ P(A wins) $\times$ Expected no. of shots, given that A wins

$+$ P(B wins) $\times$ Expected no. of shots, given that B wins

$= \alpha \cdot \frac{1}{a} + \beta \cdot \frac{1}{b}$ or $\frac{\alpha}{a} + \frac{\beta}{b}$, as required.